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WAVE MOTION IN A PLASMA BASED ON A FOKKER-PLANCK EQUATION

Alan Oppenheim

March 31, 1964

AEC Research and Development Report

NEW YORK UNIVERSITY

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ABSTRACT

We construct an isotropic approximation to the Fokker-Planck collision operator (IFP model) for the purpose of studying the small amplitude wave motion of an unbounded, fully ionized, neutral, two component plasma in a steady applied magnetic field. We solve the linearized Boltzmann's equations with Maxwell's equations and construct the dispersion relation. Because the isotropic Fokker-Planck operators describe a diffusion process in velocity space while the previously considered Krook relaxation models do not we find an infinite number of new Larmor resonances modes at arbitrary directions of propagation with respect to the applied magnetic field. The new resonances remain in the limit of long macroscopic wavelengths and small collision frequencies. At propagation parallel to the applied magnetic field we find new resonances for wavelength large and small with respect to the mean free path.

In other macroscopic ranges the Krook and IFP models give similar results. As a special example we verify the Faraday rotation formula for a particular collisional plasma

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1. Introduction

We introduce an isotropic approximation to the Fokker-Planck collision operators in Boltzmann's equations in order to study small amplitude wave motion in a fully ionized, neutral, two component plasma in a steady, applied magnetic field. With this model we find Larmor resonance modes which can not be found from theories based on the relaxation model of Bhatnagar, Gross and Krook[1] (the Krook model). We remark that so far the Krook model is the only other collision model from which one computes Larmor resonances. Moreover, these modes persist in a certain non-uniform collisionless limit.

The complexity of the collision operators either in Boltzmann integral form or in Fokker-Planck form forces one to seek simpler models to study wave motion in plasmas from a microscopic approach. The model of R.L. Liboff[2] is a generalization of the Krook model which includes the effects of ion-electron collisions. His model, therefore, has the attractive feature of giving the macroscopic conservation equations when one computes the required moments of Boltzmann's equations. A defect of all Krook models is that they do not describe a diffusion process in velocity space. The isotropic Fokker-Planck (IFP) model introduced here has two important features. It gives the macroscopic conservation equations and it describes a diffusion process in velocity space. Both properties are shared with the Boltzmann integral operators and the original Fokker-Planck operators[3]. For those effects

which are sensitive to the collision model we expect to find answers different from those found from Krook model theories. We verify that the behavior of Larmor resonances found by the IFP model is radically different from the behavior of those found by Liboff[2]. On the other hand in certain parameter ranges there are effects which are not sensitive to the collision model. An example is given by the plasma electromagnetic and magnetohydrodynamic modes in the long wavelength limit. Liboff shows that to lowest order the microscopic and macroscopic description of these modes are equivalent. The IFP results agree to lowest order with Liboff's.

We construct the dispersion relation* from the formal solution to the linearized Boltzmann's equations and Maxwell's equations. The roots of the dispersion formula are connections between frequency and wavenumber and are interpreted variously as oscillations, resonances, propagating modes and the like. In the detailed analysis of the roots we prescribe that the wavenumber vector be real and that the wavelength be large or small with respect to certain representative physical lengths. We then compute frequency in terms of wavelength and the physical parameters from the dispersion equation.

Our principal results are in the study of the Larmor resonances. These resonances are roots of the dispersion equation for which the frequency is, roughly, an integral multiple of the

---

* The terms "dispersion equation", "dispersion formula" and "dispersion relation" are used interchangeably.

Larmor frequency. There is a radical difference in the kind of resonances predicted from the IFP dispersion equation and from those predicted from the dispersion equations of the Krook model[2] or from the collisionless Boltzmann's equation[18].

From the Krook and collisionless Boltzmann's equations one finds an infinite number of Larmor resonances only at propagation exactly perpendicular to the applied magnetic field. From the IFP equation one finds an infinite number of Larmor resonances at arbitrary angles of propagation (except parallel) to the applied magnetic field. This reflects the physical property that the IFP model describes a diffusion process in velocity space while the Krook model and the collisionless Boltzmann's equation do not. There is another difference between the Krook and IFP models. At perpendicular propagation, to each Krook resonance there is an infinite number of IFP resonances separated by integral multiples of the collision frequency. This reflects the mathematical property that the IFP differential operator is unbounded with a discrete spectrum while the Krook ~~equation~~ operator is bounded.

The distinctions between the two collisional models remain in the non-uniform collisionless limit obtained by keeping collision length to wavelength fixed and small as the collision frequency becomes arbitrarily small.

A precursor to the IFP operator is the Fokker-Planck operator familiar from Brownian motion studies[4]. A. A. Logunov[5] used such an operator to study the space periodicity of gas discharges.

Somewhat later A. Lenard and I. Bernstein[6] used the same operator to study plasma oscillations. Neither of these authors considered the problem for a steady applied magnetic field. However J. Keilson, S. Schneider and A. Cantor[7,8] did study propagation in the upper atmosphere in the presence of a steady magnetic field. Their results are contained among the lowest order roots found by Liboff[2]. In none of these models were the ion-electron interactions included so that the macroscopic equations of motion of fluid dynamics are lost.

In section 2 we describe the construction of the isotropic Fokker-Planck model from the full Fokker-Planck equation. Following this the equations of motion, Boltzmann's equations and Maxwell's equations, are linearized about an absolute equilibrium state.

Secton 3 contains the formal solution to the linearized equations and the construction of the dispersion relation. At the end of this section we show how Liboff's[2] lowest order results may be obtained from our equations.

Section 4 contains the principal results of this work. These are the studies of the Larmor resonances in different limiting cases. Following the introductory section 4A we describe in 4B the analysis followed in the rest of the section except in the short wavelength limit. Here too we indicate the reason why both the collisional Krook model and the collisionless model give an infinite number of Larmor resonances only at angles of propagation exactly normal to the direction of the applied magnetic field.

Section 4C contains our study of the non-uniform limit. Section 4D contains the uniform collisionless results at angles of propagation normal to the applied magnetic field. In section 4E we consider the collisional Larmor resonances at arbitrary directions of propagation, at propagation normal to the magnetic field to compare with Liboff's results, and at propagation parallel to the magnetic field. All of these results are in the long wavelength limit, that is, long with respect to the Larmor radius and long with respect to the collision mean free path except for the uniform collisionless case where the concept of mean free path does not enter. In section 4F we present our results for propagation parallel to the magnetic field for wavelength short compared with a mean free path.

In section 5 we first calculate the effect of a plasma in a magnetic field on the electromagnetic propagation mode for propagation parallel to the applied magnetic field. The calculation is given for a set of experimental parameters suggested by Dr. Fred L. Ribe. In doing this calculation we verify the classical formula describing the Faraday rotation in the parameter range of Dr. Ribe's experiments.

We then calculate the effect of collisions on the plasma oscillation mode and compare the results with those of previous authors.

Section 6 contains the summary and conclusions of this study.

2. Equations and Linearization

A. Equations

We start with the Boltzmann's equation in which the collision processes are represented by Fokker-Planck differential operators*. These operators J have the typical form

$$J(f) = \frac{1}{2}\left(\frac{\partial^2}{\partial\xi_r \partial\xi_s}\right)(b_{rs} f) - \frac{\partial}{\partial\xi_r}(a_r f)$$

where f is the single particle distribution function, $\underset{\sim}{\xi}$** is the microscopic particle velocity with cartesian components ξ_r, b_{rs} and a_r are the diffusion and friction coefficients respectively. The coefficients b_{rs} and a_r are in turn complicated integrals in velocity space involving the distribution function of the scattering particles.

The construction (not derivation) of the Isotropic Fokker-Planck (IFP) model proceeds as follows. First we evaluate b_{rs} and a_r as if each species of scattering particles were in a local Maxwellian state. Then we evaluate the coefficients in the limit in which the particle energy is much smaller than the average energy of the scatterers. Finally we manipulate the coefficients involving ion-electron collisions until the following characteristics of the original Fokker-Planck coefficients reappear. The collision terms must vanish when both species are in the same

---

* c.f. L. Landau [9], H.Grad [10], Rosenbluth et al. [11].

** The wavy underline ($\sim$) indicates the vector quantities.

local Maxwellian state. The conservation equations for mass density, total momentum density and total energy density must be satisfied*. As a consequence an H-theorem for the gas mixture** is also satisfied. We now postulate that the coefficients are valid throughout the entire velocity space, and call the new differential operator the Isotropic Fokker-Planck operator. The results are given in equations (2.6) and (2.7) below.

Before writing the equations we need to introduce some notation and definitions. The gas under study is a fully ionized mixture of electrons and ions of one species. Ion quantities carry a caret (^) while electron quantities remain bare. The single particle distribution functions f and $\hat{f}$ are functions of time t, space $\underset{\sim}{x}$, and velocity $\underset{\sim}{\xi}$. In equations (2.1) we define the following moments of f and $\hat{f}$: The number densities in configuration space n and $\hat{n}$, the macroscopic velocities $\underset{\sim}{u}$ and $\hat{\underset{\sim}{u}}$, and the macroscopic temperatures T and $\hat{T}$.

$$
\begin{aligned}
n &= \int f d^3\xi \quad , & \hat{n} &= \int \hat{f} d^3\xi \ , \\
n\underset{\sim}{u} &= \int \underset{\sim}{\xi} f d^3 \quad , & \hat{n}\hat{\underset{\sim}{u}} &= \int \underset{\sim}{\xi} \hat{f} d^3\xi , \\
3nRT &= \int (\underset{\sim}{\xi}-\underset{\sim}{u})^2 f d^3\xi \quad , & 3\hat{n}\hat{R}\hat{T} &= \int (\underset{\sim}{\xi}-\underset{\sim}{u})^2 \hat{f} d^3\xi ,
\end{aligned}
\tag{2.1}
$$

where $R = \kappa/m$, $\hat{R} = \kappa/\hat{m}$, κ is Boltzmann's constant and m and $\hat{m}$ are the particle masses.

---

* Chapman and Cowling [12] Chapter 8.

** Chapman and Cowling [12] Chapter 4, and H. Grad [13].

In equations (2.2) we define the total number density $\bar{n}$, the total mass density $\bar{\rho}$, the total macroscopic velocity $\bar{\underset{\sim}{u}}$, the temperatures T' and $\hat{T}'$ measured with respect to $\bar{\underset{\sim}{u}}$, and the total gas temperature $\bar{T}$.

$$\bar{n} = n + \hat{n}, \qquad \bar{\rho} = mn + \hat{m}\hat{n},$$

$$\bar{\rho}\,\bar{\underset{\sim}{u}} = mn\underset{\sim}{u} + \hat{m}\hat{n}\hat{\underset{\sim}{u}}, \tag{2.2}$$

$$3nRT' = \int(\underset{\sim}{\xi}-\bar{\underset{\sim}{u}})^2 f d^3\xi, \qquad 3\hat{n}\hat{R}\hat{T}' = \int(\underset{\sim}{\xi}-\bar{\underset{\sim}{u}})^2 \hat{f} d^3\xi,$$

$$\bar{n}\,\bar{T} = nT' + \hat{n}\hat{T}'.$$

Finally the total charge density $\bar{q}$ and current density $\bar{\underset{\sim}{j}}$ are defined in terms of the signed electronic and ionic charges e and $\hat{e}$ in equations (2.3).

$$\bar{q} = ne + \hat{n}\hat{e},$$

$$\bar{\underset{\sim}{j}} = ne\underset{\sim}{u} + \hat{n}\hat{e}\hat{\underset{\sim}{u}}. \tag{2.3}$$

The equations of motion are Maxwell's equations (2.4) and the two Boltzmann's equations (2.5).

$$\nabla\cdot\underset{\sim}{E} = \bar{q}/\varepsilon_o, \qquad \nabla\cdot\underset{\sim}{B} = 0,$$

$$\nabla\times\underset{\sim}{E} = -(\partial/\partial t)\underset{\sim}{B}, \tag{2.4}$$

$$\nabla\times\underset{\sim}{B} = \mu_o\bar{\underset{\sim}{j}} + c^{-2}(\partial/\partial t)\underset{\sim}{E},$$

where $\underset{\sim}{E}$ and $\underset{\sim}{B}$ are the electric and magnetic field vectors

respectively, ε_o and μ_o are the dielectric constant and permeability of free space respectively, and $c = (\varepsilon_o\mu_o)^{-\frac{1}{2}}$ is the speed of light. We use rationalized MKS units throughout.

$$\frac{\partial f}{\partial t} + \underset{\sim}{\xi}\cdot\nabla f + \frac{e}{m}(\underset{\sim}{E} + \underset{\sim}{\xi}\times\underset{\sim}{B})\cdot\nabla_\xi f = \left(\frac{\partial f}{\partial t}\right)_{coll} ,$$

$$\frac{\partial \hat{f}}{\partial t} + \underset{\sim}{\xi}\cdot\nabla \hat{f} + \left(\frac{\hat{e}}{m}\right)(\underset{\sim}{E} + \underset{\sim}{\xi}\times\underset{\sim}{B})\cdot\nabla_\xi \hat{f} = \left(\frac{\partial \hat{f}}{\partial t}\right)_{coll} , \tag{2.5}$$

where ∇_ξ is the gradient operator in velocity space, and the collision terms $\left(\frac{\partial f}{\partial t}\right)_{coll}$ and $\left(\frac{\partial \hat{f}}{\partial t}\right)_{coll}$ are now those of the Isotropic Fokker-Planck model:

$$\left(\frac{\partial f}{\partial t}\right)_{coll} = J_1(f) + J_2(f),$$

$$\left(\frac{\partial \hat{f}}{\partial t}\right)_{coll} = \hat{J}_1(\hat{f}) + \hat{J}_2(\hat{f}), \tag{2.6}$$

where

$$J_1(f) = \nu_1 \frac{\partial}{\partial \xi_r}\left(RT\frac{\partial f}{\partial \xi_r} + (\xi_r - u_r)f\right),$$

$$J_2(f) = \nu_2 \frac{\partial}{\partial \xi_r}\left(R\overline{T}\frac{\partial f}{\partial \xi_r} + (\xi_r - \overline{u}_r)f\right),$$

$$\hat{J}_1(\hat{f}) = \hat{\nu}_1 \frac{\partial}{\partial \xi_r}\left(\hat{R}\hat{T}\frac{\partial \hat{f}}{\partial \xi_r} + (\xi_r - \hat{u}_r)\hat{f}\right), \tag{2.7}$$

$$\hat{J}_2(\hat{f}) = \nu_2 \frac{\partial}{\partial \xi_r}\left(\hat{R}\overline{T}\frac{\partial \hat{f}}{\partial \xi_r} + (\xi_r - \overline{u}_r)\hat{f}\right).$$

The collision frequencies ν_1, $\hat{\nu}_1$ and ν_2 are unknown functions of space and time. That they are unknown is the penalty exacted

for simplifying the Fokker-Planck coefficients. The collision terms have the following properties. They vanish when f and $\hat{f}$ are in the same local Maxwellian state, that is when

$$f = n(2\pi RT)^{-3/2} \exp(-[\underset{\sim}{\xi}-\underset{\sim}{u}]^2/2RT) ,$$

$$\hat{f} = \hat{n}(2\pi\hat{R}\hat{T})^{-3/2} \exp(-[\underset{\sim}{\xi}-\underset{\sim}{u}]^2/2\hat{R}\hat{T}) ,$$

$$\underset{\sim}{u} = \hat{\underset{\sim}{u}} \quad \text{and} \quad T = \hat{T} .$$

The macroscopic conservation equations follow from the identities (2.8).

$$\begin{aligned}
&\int J_1(f)\begin{pmatrix} 1 \\ \underset{\sim}{\xi} \\ \xi^2 \end{pmatrix} d^3\xi = 0, \\
&\int \hat{J}_1(\hat{f})\begin{pmatrix} 1 \\ \underset{\sim}{\xi} \\ \xi^2 \end{pmatrix} d^3\xi = 0 \\
&\int J_2(f)d^3\xi = \int \hat{J}_2(\hat{f})d^3\xi = 0, \\
&m\int \underset{\sim}{\xi}\, J_2(f)d^3\xi + \hat{m}\int \underset{\sim}{\xi}\, \hat{J}_2(\hat{f})d^3\xi = 0, \\
&\frac{m}{2}\int \xi^2 J_2(f)d^3\xi + \frac{\hat{m}}{2}\int \xi^2 \hat{J}_2(\hat{f})d^3\xi = 0.
\end{aligned} \tag{2.8}$$

In addition one can derive an H-theorem. We define $H = \int f \log f \, d^3\xi + \int \hat{f} \log \hat{f} \, d^3\xi$. The contributions to $\frac{\partial H}{\partial t}$ coming

from the collision terms $\left(\frac{\partial H}{\partial t}\right)_{coll} \leq 0$, with equality when f and $\hat{f}$ are in the same local Maxwellian state. Finally the IFP operators describe a diffusion process in velocity space, a property shared with the Boltzmann integral operators and the full Fokker-Planck operators[3].

B. Linearization

We linearize the equations of motion about the equilibrium solutions f_o and $\hat{f}_o$ in the rest frame of the total gas. This means that all of the quantities: the distribution functions, their moments, and the field variables are written in the form

$$X \rightarrow X_o + X \,, \tag{2.9}$$

where the arrow signifies that wherever X appears in the equations and definitions it is replaced with the sum of its equilibrium value X_o and its perturbation value X. Products of two or more perturbed quantities are dropped systematically. The equilibrium solutions are the absolute maxwellians

$$\begin{aligned} f_o &= n_o(2\pi C^2)^{-3/2} \exp(-\xi^2/2C^2), \\ \hat{f}_o &= \hat{n}_o(2\pi \hat{C}^2)^{-3/2} \exp(-\xi^2/2\hat{C}^2), \end{aligned} \tag{2.10}$$

where $C^2 = RT_o$, $\hat{C}^2 = \hat{R}\hat{T}_o$, $T_o = \hat{T}_o$, and the quantities n_o, $\hat{n}_o$ and T_o are constants. The property that the equilibrium plasma be neutral means that $|e|n_o = |\hat{e}|\hat{n}_o$.

We suppose for simplicity that the positive particles are singly ionized so that from this point on $n_o = \hat{n}_o$. Once we have linearized there is no difference between T and T' and between $\hat{T}$ and $\hat{T}'$ since they differ by second order quantities in the perturbed variables. The full list of variables appears in equations (2.11).

$$
\begin{aligned}
&f \rightarrow n_o(g_o + g) &&, \quad \hat{f} \rightarrow n_o(\hat{g}_o + \hat{g}), \\
&n \rightarrow n_o + n &&, \quad \hat{n} \rightarrow n_o + \hat{n}, \quad \bar{\rho} \rightarrow \bar{\rho}_o + \bar{\rho}, \\
&\underset{\sim}{u} \rightarrow 0 + \underset{\sim}{u} &&, \quad \hat{\underset{\sim}{u}} \rightarrow 0 + \hat{\underset{\sim}{u}}, \quad \bar{\underset{\sim}{u}} \rightarrow 0 + \hat{\underset{\sim}{u}}, \\
&T \rightarrow T_o + T &&, \quad \hat{T} \rightarrow T_o + \hat{T}, \quad \bar{T} \rightarrow T_o + \bar{T}, \\
&\bar{q} \rightarrow 0 + \bar{q} &&, \quad \bar{\underset{\sim}{j}} \rightarrow 0 + \bar{\underset{\sim}{j}}, \\
&\underset{\sim}{E} \rightarrow 0 + \underset{\sim}{E} &&, \quad \underset{\sim}{B} \rightarrow \underset{\sim}{B}_o + \underset{\sim}{B},
\end{aligned}
\tag{2.11}
$$

where $n_o g_o = f_o$, $n_o \hat{g}_o = \hat{f}_o$, the perturbed $f = n_o g$, the perturbed $\hat{f} = n_o \hat{g}$, and $\underset{\sim}{B}_o$ is the constant applied magnetic field. In terms of g the moments are given by

$$
\begin{aligned}
n &= n_o \int g d^3\xi , \\
\underset{\sim}{u} &= \int \underset{\sim}{\xi} g d^3\xi , \\
(T/T_o) &= \int [\xi^2/3C^2 - 1] g d^3\xi ,
\end{aligned}
\tag{2.12}
$$

and similarly for the ion quantities.

Maxwell's equations (2.4) remain unchanged in form. Boltzmann's equations (2.5) and the collision terms (2.6) and

(2.7) become

$$\frac{\partial g}{\partial t} + \underset{\sim}{\xi}\cdot\nabla g + \underset{\sim}{\xi}\times\underset{\sim}{\Omega}\cdot\nabla_{\xi} g - J_o(g) = \left\{\frac{\underset{\sim}{a}\cdot\underset{\sim}{\xi}}{C^2} + b\left(\frac{\xi^2}{C^2} - 3\right)\right\} g_o$$

$$\frac{\partial \hat{g}}{\partial t} + \underset{\sim}{\xi}\cdot\nabla \hat{g} + \underset{\sim}{\xi}\times\underset{\sim}{\Omega}\cdot\nabla_{\xi} \hat{g} - \hat{J}_o(\hat{g}) = \left\{\frac{\hat{\underset{\sim}{a}}\cdot\underset{\sim}{\xi}}{\hat{C}^2} + \hat{b}\left(\frac{\xi^2}{\hat{C}^2} - 3\right)\right\} \hat{g}_o, \qquad (2.13)$$

where

$$J_o(g) = \nu \frac{\partial}{\partial \xi_r}\left(C^2 \frac{\partial g}{\partial \xi_r} + \xi_r g\right),$$

$$\hat{J}_o(\hat{g}) = \hat{\nu} \frac{\partial}{\partial \xi_r}\left(\hat{C}^2 \frac{\partial \hat{g}}{\partial \xi_r} + \xi_r \hat{g}\right),$$

$$\underset{\sim}{a} = \frac{e}{m}\underset{\sim}{E} + \nu_1 \underset{\sim}{u} + \nu_2 \underset{\sim}{\bar{u}},$$

$$\hat{\underset{\sim}{a}} = \frac{\hat{e}}{\hat{m}}\underset{\sim}{E} + \hat{\nu}_1 \hat{\underset{\sim}{u}} + \nu_2 \underset{\sim}{\bar{u}},$$

$$b = (\nu_1 T + \nu_2 \bar{T})/T_o,$$

$$\hat{b} = (\hat{\nu}_1 \hat{T} + \nu_2 \bar{T})/T_o,$$

$$\underset{\sim}{\Omega} = e\,\underset{\sim}{B}_o/m, \quad \hat{\underset{\sim}{\Omega}} = \hat{e}\,\underset{\sim}{B}_o/\hat{m},$$

$$\nu = \nu_1 + \nu_2, \quad \hat{\nu} = \hat{\nu}_1 + \nu_2$$

The collision frequencies ν_1, $\hat{\nu}_1$ and ν_2 are unknown constants in this theory. None the less, it is useful to have expressions for them which indicate dependence on physical parameters such as mass and temperature. For this reason we give in equations (2.14) expressions for ν_1 and $\hat{\nu}_1$ which come

directly from the original Fokker-Planck coefficients, and an expression for ν_2 which derives from the time of equipartition of energy discussed, for example, in chapter 5 of L. Spitzer's book[14].

$$\nu_1 = \frac{e^4 n_o \log \Lambda}{3m^{\frac{1}{2}} \varepsilon_o^2 (2\pi\kappa T_o)^{3/2}} ,$$

$$\hat{\nu}_1 = \left(\frac{m}{\hat{m}}\right)^{\frac{1}{2}} \nu_1 , \qquad (2.14)$$

$$\nu_2 = \frac{2m\hat{m}^{\frac{1}{2}}}{(m+\hat{m})^{3/2}} \nu_1 ,$$

where

$$\Lambda = 6\pi \left(\frac{2}{n_o}\right)^{\frac{1}{2}} \left(\frac{\kappa T_o \varepsilon_o}{e^2}\right)^{3/2} .$$

We assume that equations (2.14) give the correct dependence of the collision frequencies on the ion and electron masses.

3. Construction of the Dispersion Relation

A. Program of the construction

This brief outline of the various steps which lead from the equations of motion to the final dispersion relation is intended partly as a guide to the details and partly as a way of defining what we mean by a dispersion relation in terms of its construction.

Subsection B contains the formal solution to the Boltzmann's equations (2.13). We illustrate with the equation for g which can be written

$$L(t,\underset{\sim}{x},\underset{\sim}{\xi})g = R\, g_o \quad .$$

L is the differential operator; R is a linear combination of $\underset{\sim}{E}$ and the unknown moments $\underset{\sim}{u}$, $\hat{\underset{\sim}{u}}$, T, and $\hat{T}$. Under the Laplace-Fourier transformation $\int_0^\infty e^{-st}dt \int e^{i\,\underset{\sim}{k}\cdot\underset{\sim}{x}}\, d^3x \int e^{i\,\underset{\sim}{w}\cdot\underset{\sim}{\xi}}\, d^3\xi$ the equation becomes

$$L(s,\underset{\sim}{k},\underset{\sim}{w})g = R\, g_o + g_i$$

where g_i is the initial value of g, and L is now a first order linear partial differential equation in $\underset{\sim}{w}$. We invert the equation and get

$$g(s,\underset{\sim}{k},\underset{\sim}{w}) = L^{-1}(R\, g_o + g_i) \ .$$

This means that g is known in terms of $\underset{\sim}{E}$ and the moments $\underset{\sim}{u}$, $\hat{\underset{\sim}{u}}$, T and $\hat{T}$. There is a similar equation for $\hat{g}$. Maxwell's equations and the equations of continuity enable~~s~~ us to write

$\underset{\sim}{E}(s,\underset{\sim}{k})$ in terms of the moments $\underset{\sim}{u}(s,\underset{\sim}{k})$ and $\hat{\underset{\sim}{u}}(s,\underset{\sim}{k})$ and the initial values of the field quantities. The equations for g and $\hat{g}$ are not closed. That is, g and $\hat{g}$ are given in terms of their unknown moments $\underset{\sim}{u}(s,\underset{\sim}{k})$, $\hat{\underset{\sim}{u}}(s,\underset{\sim}{k})$, $T(s,\underset{\sim}{k})$ and $\hat{T}(s,\underset{\sim}{k})$. We cannot know g and $\hat{g}$ unless we know those moments. In subsection C we close the equations and solve formally for the moments. This procedure gives us the dispersion relation which is our goal.

The construction given in subsection C rests on the identities

$$n(s,\underset{\sim}{k}) \sim g(s,\underset{\sim}{k},0) \ ,$$

$$\underset{\sim}{u}(s,\underset{\sim}{k}) \sim \nabla_{w} g(s,\underset{\sim}{k},\underset{\sim}{w})\Big|_{\underset{\sim}{w}=0} \ ,$$

etc. With these relations and the corresponding ones for the ion variables we can form the relation

$$A(s,\underset{\sim}{k})\ U(s,\underset{\sim}{k}) = V(s,\underset{\sim}{k})$$

where A is now a known matrix, U is a column vector consisting of $\underset{\sim}{E}$ and the moments $\underset{\sim}{u}, \hat{\underset{\sim}{u}}$, T and $\hat{T}$, and V is a column vector containing the initial data. The dispersion relation is

$$\det A(s,\underset{\sim}{k}) = 0.$$

The roots of the dispersion relation are equations connecting s and $\underset{\sim}{k}$. It is clear that knowledge of the analytic properties of the initial data will tell us which of the roots of the dispersion relation are relevant to a given problem[15]. We will neglect this aspect of the problem and concentrate on the dis-

persion relation which contains all possible roots.

B. Formal Solution to the Linearized Boltzmann's Equations

Under the transformation

$$\int_0^\infty e^{-st}\, dt \int e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}\, d^3x \int e^{i\underset{\sim}{w}\cdot\underset{\sim}{\xi}}\, d^3\xi \tag{3.1}$$

the first of equations (2.13) becomes

$$\begin{aligned}(s + \nu c^2 w^2)g(s,\underset{\sim}{k},\underset{\sim}{w}) + (\nu\underset{\sim}{w} - \underset{\sim}{k} + \underset{\sim}{w}\times\underset{\sim}{\Omega})\cdot\nabla_w g(s,\underset{\sim}{k},\underset{\sim}{w}) = \\ (i\,\underset{\sim}{a}(s,\underset{\sim}{k})\cdot w - c^2w^2 b(s,\underset{\sim}{k}))e^{-c^2w^2/2} + g_i(\underset{\sim}{k},\underset{\sim}{w}),\end{aligned} \tag{3.2}$$

where $g_i(\underset{\sim}{k},\underset{\sim}{w})$ is the transform of the initial value of g at time $t = 0$. The equation for $\hat{g}$ is similar with the caret ($\hat{}$) over the appropriate quantities. Although we will no longer carry the initial values in the transformed Boltzmann's and Maxwell's equations their presence is implied throughout the development.

To integrate equation (3.2) we look for solutions to

$$\frac{d\,\underset{\sim}{w}(\tau)}{d\tau} = \nu\underset{\sim}{w}(\tau) - \underset{\sim}{k} + \underset{\sim}{w}(\tau)\times\underset{\sim}{\Omega} \tag{3.3}$$

subject to

$$\underset{\sim}{w}(0) = \underset{\sim}{w}\,. \tag{3.4}$$

If $\underset{\sim}{w}(\tau)$ satisfies (3.3) we can write equation (3.2) in the form

$$(s+\nu C^2 w^2(\tau))g + \frac{dg}{d\tau} = (i\,\underset{\sim}{a}\cdot\underset{\sim}{w}(\tau) - C^2 w^2(\tau) b)e^{-C^2 w^2(\tau)/2} \quad . \qquad (3.5)$$

Equation (3.5) can be integrated easily. The condition (3.4) means that we can integrate equation (3.5) for all $\underset{\sim}{w}$.

Solutions to (3.3) are of the form

$$\underset{\sim}{w}(\tau) = \underset{\sim}{\mu}(1) + e^{\nu\tau}(\underset{\sim}{\mu}(2) + e^{\sigma\tau}\underset{\sim}{\mu}(3))$$

where $\underset{\sim}{\mu}(1)$, $\underset{\sim}{\mu}(2)$, and $\underset{\sim}{\mu}(3)$ are vectors independent of τ, and σ is a constant to be determined. We impose condition (3.4) and get, after some straightforward vector manipulation,

$$\underset{\sim}{w}(\tau) = \underset{\sim}{\mu}(1) + e^{\nu\tau}\left\{(\underset{\sim}{w}-\underset{\sim}{\mu}(1))\cdot\underset{\sim}{\Omega}/\Omega^2 + e^{i\Omega\tau}\underset{\sim}{\gamma}_+ + e^{-i\Omega\tau}\underset{\sim}{\gamma}_-\right\} \qquad (3.6)$$

where

$$\underset{\sim}{\mu}(1) = [\underset{\sim}{k}\nu^2 + \nu\underset{\sim}{\Omega}\times\underset{\sim}{k} + \underset{\sim}{\Omega}(\underset{\sim}{\Omega}\cdot\underset{\sim}{k})]/\nu(\Omega^2+\nu^2) \quad ,$$

$\Omega = e\,B_o/m$ carries the sign of the electron charge e,

and

$$\underset{\sim}{\gamma}_\pm = \tfrac{1}{2}\left[\frac{\underset{\sim}{\Omega}\times[(\underset{\sim}{w}-\underset{\sim}{\mu}(1))\times\underset{\sim}{\Omega}]}{\Omega^2} \pm \frac{1}{i}\,\frac{(\underset{\sim}{w}-\underset{\sim}{\mu}(1))\times\underset{\sim}{\Omega}}{\Omega}\right] .$$

The integrating factor for equation (3.5) is $\exp\int_{\tau_o}^{\tau}[s+\nu C^2 w^2(\tau')]d\tau'$. The lower limit τ_o is arbitrary and drops out in the subsequent manipulations. We multiply equation (3.5) by the integrating factor and integrate from $\tau = -\infty$ to $\tau = 0$. The lower limit on the left hand side vanishes in the

range of validity of the original transformation (3.1). The result, after some trivial changes in the variables of integration, is

$$g(s,\underset{\sim}{k},\underset{\sim}{w}) = \int_0^\infty dy\; e^{\phi(y,\underset{\sim}{w})}\; H(y,\underset{\sim}{w}), \tag{3.7}$$

where $\phi(y,\underset{\sim}{w})$ and $H(y,\underset{\sim}{w})$ are defined in equations (3.8) and (3.9) respectively.

$$\begin{aligned}\phi(y,w) = &- (s+\nu C^2\mu^2_{(1)})y - \frac{C^2w^2}{2} \\ &- C^2(\underset{\sim}{w}-\underset{\sim}{\mu}(1))\cdot\left\{\frac{\underset{\sim}{\Omega}(\underset{\sim}{\Omega}\cdot\underset{\sim}{\mu}(1))}{\Omega^2}(1-e^{-\nu y}) - \underset{\sim}{\Gamma}_+\, e^{i\psi}(1-e^{-(\nu+i\Omega)y})\right. \\ &\left. - \underset{\sim}{\Gamma}_-\, e^{-i\psi}(1-e^{-(\nu-i\Omega)y})\right\},\end{aligned} \tag{3.8}$$

where

$$e^{\pm i\psi} = -\frac{(\nu\mp i\Omega)^2}{\nu^2+\Omega^2},$$

and

$$\underset{\sim}{\Gamma}_\pm = \frac{1}{2}\left[\frac{\underset{\sim}{\Omega}\times(\underset{\sim}{\mu}(1)\times\underset{\sim}{\Omega})}{\Omega^2} \pm \frac{1}{i}\frac{\underset{\sim}{\Omega}\times\underset{\sim}{\mu}(1)}{\Omega}\right].$$

$$\begin{aligned}H(y,\underset{\sim}{w}) = i\Bigg\{&\underset{\sim}{\mu}(1)\cdot\underset{\sim}{a} + (\underset{\sim}{w}-\underset{\sim}{\mu}(1))\cdot\Bigg[\underset{\sim}{\Omega}\frac{(\underset{\sim}{\Omega}\cdot\underset{\sim}{a})}{\Omega^2}e^{-\nu y} + \underset{\sim}{A}_+\, e^{-(\nu+i\Omega)y} \\ &+ \underset{\sim}{A}_-\, e^{-(\nu-i\Omega)y}\Bigg]\Bigg\} - w^2(-y)C^2b,\end{aligned} \tag{3.9}$$

where $\underset{\sim}{A}_\pm = \frac{1}{2}\left[\frac{\underset{\sim}{\Omega}\times(\underset{\sim}{a}\times\underset{\sim}{\Omega})}{\Omega^2} \pm \frac{1}{i}\frac{\underset{\sim}{\Omega}\times\underset{\sim}{a}}{\Omega}\right]$,

$$w^2(-y) = \mu^2_{(1)} + 2(\underset{\sim}{w}-\underset{\sim}{\mu}_{(1)})\cdot\left[\frac{\underset{\sim}{\Omega}(\underset{\sim}{\Omega}\cdot\underset{\sim}{\mu}_{(1)})e^{-\nu y}}{\Omega^2} + \underset{\sim}{\Gamma}_+ e^{-(\nu+i\Omega)y}\right.$$

$$\left. + \underset{\sim}{\Gamma}_- e^{-(\nu-i\Omega)y}\right] + (\underset{\sim}{w}-\underset{\sim}{\mu}_{(1)})^2 e^{-2\nu y} ,$$

and

$$\mu^2_{(1)} = (k^2\nu^2+(\underset{\sim}{k}\cdot\underset{\sim}{\Omega})^2)/\nu^2(\nu^2+\Omega^2).$$

Equation (3.7) is a formal solution to Boltzmann's equation. That is, $g(s,\underset{\sim}{k},\underset{\sim}{w})$ is given linearly in terms of the unknown moments $n(s,\underset{\sim}{k})$, $\hat{n}(s,\underset{\sim}{k})$, $\underset{\sim}{u}(s,\underset{\sim}{k})$, $\hat{\underset{\sim}{u}}(s,\underset{\sim}{k})$, $T(s,\underset{\sim}{k})$ and $\hat{T}(s,\underset{\sim}{k})$, and the unknown electric field $\underset{\sim}{E}(s,\underset{\sim}{k})$. The coefficients of the moments and $\underset{\sim}{E}$ are integrals which are known in principle. The same statements apply to $\hat{g}(s,\underset{\sim}{k},\underset{\sim}{w})$. Consequently <u>all</u> of the moments of g and $\hat{g}$ are also expressed as linear functions of $n,\hat{n},\underset{\sim}{u},\hat{\underset{\sim}{u}},T,\hat{T}$ and $\underset{\sim}{E}$.

In the remainder of this section we reduce the problem to that of solving for $\underset{\sim}{u},\hat{\underset{\sim}{u}}$, $T,\hat{T}$ and $\underset{\sim}{E}$, and, in so doing, construct the dispersion relation.

C. Construction of the Dispersion Relation

We construct the moments in $(s,\underset{\sim}{k})$ space directly from the definitions (2.12) and the identity

$$g(s,\underset{\sim}{k},\underset{\sim}{w}) = \int g(s,\underset{\sim}{k},\underset{\sim}{\xi})e^{i\underset{\sim}{w}\cdot\underset{\sim}{\xi}}d^3\xi \ . \tag{3.10}$$

$$n(s,\underset{\sim}{k})/n_o = \int g(s,\underset{\sim}{k},\underset{\sim}{\xi})d^3\xi = g(s,k,w)\Big|_{w=0},$$

$$\underset{\sim}{u}(s,\underset{\sim}{k}) = \int \underset{\sim}{\xi} g(s,\underset{\sim}{k},\underset{\sim}{\xi}) d^3\xi = \frac{1}{i} \nabla_w \, g(s,\underset{\sim}{k},\underset{\sim}{w})\Big|_{w=0} , \tag{3.11}$$

$$T(s,\underset{\sim}{k})/T_o = \int \left(\frac{\xi^2}{3C^2} - 1\right) g(s,\underset{\sim}{k},\underset{\sim}{\xi}) d^3\xi$$

$$= - \left((3C^2)^{-1} \nabla_w^2 g(s,\underset{\sim}{k}\underset{\sim}{w}) + g(s,\underset{\sim}{k}\underset{\sim}{w})\right)\Big|_{w=0} .$$

The moments $\hat{n}$, $\hat{\underset{\sim}{u}}$, and $\hat{T}$ are computed from $\hat{g}$ in exactly the same way. The results for g and $\hat{g}$ are parallel, differing only in the presence of the caret (^) over the appropriate ion variables. All that we need to complete the system of equations is a relation between $\underset{\sim}{E}$ and the moments of g and $\hat{g}$. This relation comes from the transformed Maxwell's equations (3.12). The initial values are not written but implied.

$$\underset{\sim}{k}\cdot\underset{\sim}{B} = 0 \quad , \qquad \underset{\sim}{k}\cdot\underset{\sim}{E} = i\bar{q}/\varepsilon_o ,$$

$$i\underset{\sim}{k}\times\underset{\sim}{E} = s\underset{\sim}{B} \quad , \qquad -i\underset{\sim}{k}\times\underset{\sim}{B} = \mu_o\underset{\sim}{\bar{j}} + s\underset{\sim}{E}/c^2 \quad . \tag{3.12}$$

We eliminate $\underset{\sim}{B}$ from the equations (3.12) and get the required relation between $\underset{\sim}{E}$ and the moments of g and $\hat{g}$.

$$\underset{\sim}{E} = \varepsilon_o^{-1}(k^2c^2 + s^2)^{-1} \, (i \, \underset{\sim}{k} \, c^2\bar{q} - s\underset{\sim}{\bar{j}}) \; . \tag{3.13}$$

The equation of continuity, which is a consequence of Maxwell's equations, is

$$s\bar{q} - i \, \underset{\sim}{k}\cdot\underset{\sim}{\bar{j}} = 0 \; . \tag{3.14}$$

We use equations (2.3), (2.11) and (3.14) to write $\underset{\sim}{E}$ in terms of $\hat{\underset{\sim}{u}}$ and $\underset{\sim}{u}$.

$$\underset{\sim}{E} = - \frac{|e| n_o}{\varepsilon_o s} \left[\frac{s^2(\hat{\underset{\sim}{u}}-\underset{\sim}{u}) + \underset{\sim}{p}(\underset{\sim}{p}\cdot(\hat{\underset{\sim}{u}}-\underset{\sim}{u}))}{s^2 + p^2} \right] \tag{3.15}$$

where

$$\underset{\sim}{p} = \underset{\sim}{k}c.$$

The vector $\underset{\sim}{p}$, which comes from Maxwell's equations, is treated differently from the vectors $\underset{\sim}{K} = \underset{\sim}{k}C$ and $\hat{\underset{\sim}{K}} = \underset{\sim}{k}\hat{C}$, which come from the Boltzmann's equations. One reason for such separate treatment is that in some limits where K and $\hat{K}$ are small p itself could be finite or large.

The evaluation of the moments given by equations (3.11) simplifies in the rectangular coordinate system in which $\underset{\sim}{B}_o$ is directed along one of the axes. We represent any vector $\underset{\sim}{X}$ by its three coordinate components as follows:

$$\underset{\sim}{X} = (X_1, X_2, X_3) .$$

In this coordinate system

$$\underset{\sim}{B}_o = B_o(0,0,1) .$$

With no loss in generality we specify that $\underset{\sim}{k}$ lie in the 1-3 plane:

$$\underset{\sim}{k} = k(\sin\theta,\ 0,\ \cos\theta)$$

where θ is the angle between the directions of $\underset{\sim}{k}$ and $\underset{\sim}{B}_o$.

The system is now completely symmetric about the $\underset{\sim}{B}_o$ axis. From this point on we replace the X_1 and X_2 components of any vector $\underset{\sim}{X}$ with the combination X_+ and X_-, where

$$X_{\pm} = X_1 \pm iX_2 .$$

After performing the operations indicated in equations (3.11) we get the equations for the moments of g.

$$\begin{aligned}
u_3 &= a_3 I_{33} + a_+ \left(\frac{\sin\theta\cos\theta}{2}\right) I_{3+} + a_- \left(\frac{\sin\theta\cos\theta}{2}\right) I_{3-} \\
&\quad - iCb\cos\theta\, I_{3T} , \\
u_+ &= a_3(\sin\theta\cos\theta) I_{3+} + a_+ I_{++} + a_- \left(\frac{\sin^2\theta}{2}\right) I_{+-} \\
&\quad - iCb\sin\theta\, I_{+T} , \\
u_- &= a_3(\sin\theta\cos\theta) I_{3-} + a_+ \left(\frac{\sin^2\theta}{2}\right) I_{+-} + a_- I_{--} - iCb\sin\theta\, I_{-T} , \\
-3C\frac{T}{T_o} &= ia_3\cos\theta\, I_{3T} + ia_+ \frac{\sin\theta}{2} I_{+T} + ia_- \frac{\sin\theta}{2} I_{-T} - CbI_{TT} .
\end{aligned} \tag{3.16}$$

The various integrals I_{ij} are given in Appendix A. The four equations (3.16) with the corresponding four equations for the ion quantities and the equations for E_3, E_+ and E_- given by (3.15) are a closed set of equations.

When we use equations (3.15) to eliminate $\underset{\sim}{E}$ we end up with eight equations for the eight unknowns u_3, u_+, u_-, T, $\hat{u}_3$, $\hat{u}_+$, $\hat{u}_-$, and $\hat{T}$.

The remaining substitutions in $\underset{\sim}{a}$, $\hat{\underset{\sim}{a}}$, b and $\hat{b}$ (given by equations (2.13)) are

$$\bar{\underset{\sim}{u}} = (m\underset{\sim}{u} + \hat{m}\hat{\underset{\sim}{u}})/(m+\hat{m})$$

and (3.17)

$$\bar{T} = (T+\hat{T})/2 \ .$$

Equations (3.17) come from equations (2.2) and (2.11).

The final form of the equations is the matrix equation

$$AU = V \tag{3.18}$$

where U is a column vector with the components in descending order: u_3, u_+, u_-, T/T_o, $\hat{u}_3$, $\hat{u}_+$, $\hat{u}_-$, $\hat{T}/T_o$. The vector V contains the initial values, and A is an 8×8 matrix of the form

$$A = \begin{pmatrix} D_1 & D_2 \\ \hat{D}_2 & \hat{D}_1 \end{pmatrix} . \tag{3.19}$$

The matrices D_1, D_2, $\hat{D}_1$ and $\hat{D}_2$ are each 4×4. Before writing down these matrices we need to define certain quantities which appear in them.

$$M = m\hat{m}/(m+\hat{m}) = \text{reduced mass}, \tag{3.20}$$

$$\omega_o^2 = e^2 n_o/(\varepsilon_o M) = \text{plasma frequency squared},$$

$$\underset{\sim}{K} = k\underset{\sim}{C} \ , \quad \hat{\underset{\sim}{K}} = k\hat{\underset{\sim}{C}} \ ,$$

$$\nu_2^* = \nu_2 M/m \ , \quad \hat{\nu}_2^* = \nu_2 M/\hat{m} \ ,$$

$$\nu^* = \nu - \nu_2^* \ , \quad \hat{\nu}^* = \hat{\nu} - \hat{\nu}_2^* \ ,$$

$$\chi = \left(\frac{M}{m}\right) \frac{\omega_o^2 s}{s^2+p^2} \quad , \quad \hat{\chi} = \left(\frac{M}{\hat{m}}\right) \frac{\omega_o^2 s}{s^2+p^2} \quad ,$$

$$\eta = \left(\frac{M}{m}\right) \frac{\omega_o^2 p^2}{s(s^2+p^2)} \quad , \quad \hat{\eta} = \left(\frac{M}{\hat{m}}\right) \frac{\omega_o^2 p^2}{s(s^2+p^2)} \quad ,$$

$$\alpha = \nu^* - \chi \quad , \quad \hat{\alpha} = \hat{\nu}^* - \hat{\chi}$$

$$\beta = \nu_2^* + \chi \quad , \quad \hat{\beta} = \hat{\nu}^*_2 + \hat{\chi} \quad .$$

The matrices D_1 and D_2 are given below in equations (3.21) and (3.22) respectively. In them there are the additional integrals H_3, H_+, H_-, and H_T which are defined in appendix A along with the integrals I_{33}, I_{3+}, etc. In order to get $\hat{D}_1$ and $\hat{D}_2$ from D_1 and D_2 respectively we need only place carets above the appropriate quantities. Thus $\hat{D}_1$ is obtained from D_1 by placing carets over α, η, C, ν and the integrals. A caret over the appropriate integral, say $\hat{I}_{33}$, means K, ν, Ω, μ and ψ go to $\hat{K}$, $\hat{\nu}$, $\hat{\Omega}$, $\hat{\mu}$ and $\hat{\psi}$ in the integral I_{33} appearing in appendix A.

$$D_1 = \begin{bmatrix} (-1+\alpha I_{33}-\eta H_3\cos^2\theta) & \frac{1}{2}(\alpha I_{3+}-\eta H_3)\sin\theta\cdot\cos\theta & \frac{1}{2}(\alpha I_{3-}-\eta H_3)\sin\theta\cdot\cos\theta & -iC(\nu-\frac{1}{2}\nu_2)I_{3T}\cos\theta \\ (\alpha I_{3+}-\eta H_+)\sin\theta\cdot\cos\theta & (-1+\alpha I_{++} - \frac{\eta}{2} H_+\sin^2\theta) & \frac{1}{2}(\alpha I_{+-}-\eta H_+)\sin^2\theta & -iC(\nu-\frac{1}{2}\nu_2)I_{+T}\sin\theta \\ (\alpha I_{3-}-\eta H_-)\sin\theta\cdot\cos\theta & \frac{1}{2}(\alpha I_{+-}-\eta H_T)\sin^2\theta & (-1+\alpha I_{--} - \frac{\eta}{2} H_-\sin^2\theta) & -iC(\nu-\frac{1}{2}\nu_2)I_{-T}\sin\theta \\ i(\alpha I_{3T}-\eta H_T)\cos\theta & \frac{i}{2}(\alpha I_{+T}-\eta H_T)\sin\theta & \frac{i}{2}(\alpha I_{-T}-\eta H_T)\sin\theta & C(3-(\nu-\frac{1}{2}\nu_2)I_{TT}) \end{bmatrix} \tag{3.21}$$

$$D_2 = \begin{bmatrix} (\beta I_{33}+\eta H_3\cos^2\theta) & \frac{1}{2}(\beta I_{3+}+\eta H_3)\sin\theta\cdot\cos\theta & \frac{1}{2}(\beta I_{3-}+\eta H_3)\sin\theta\cdot\cos\theta & -\frac{iC\nu_2}{2} I_{3T}\cos\theta \\ (\beta I_{3+}+\eta H_+)\sin\theta\cdot\cos\theta & (\beta I_{++}+\frac{\eta}{2} H_+\sin^2\theta) & \frac{1}{2}(\beta I_{+-}+\eta H_+)\sin^2\theta & -\frac{iC\nu_2}{2} I_{+T}\sin\theta \\ (\beta I_{3-}+\eta H_-)\sin\theta\cdot\cos\theta & \frac{1}{2}(\beta I_{+-}+\eta H_-)\sin^2\theta & (\beta I_{--}+\frac{\eta}{2} H_-\sin^2\theta) & -\frac{iC\nu_2}{2} I_{-T}\sin\theta \\ i(\beta I_{3T}+\eta H_T)\cos\theta & \frac{i}{2}(\beta I_{+T}+\eta H_T)\sin\theta & \frac{i}{2}(\beta I_{-T}+\eta H_T)\sin\theta & -\frac{C\nu_2}{2} I_{TT} \end{bmatrix} \tag{3.22}$$

The dispersion relation is

$$\det A = 0 \,. \tag{3.23}$$

The 8×8 determinant of A can be reduced to a more compact 4×4 determinant with the use of standard transformations such as may be found in A.C. Aitken's book[16]. It is, however, easier to take limits with det A itself so that our future manipulations start from equation (3.23).

The elements of A are complicated functions of s, k, θ, the physical parameters $\Omega, \hat{\Omega}$, $m, \hat{m}$, $\nu_1, \hat{\nu}_2$ and ω_o^2, the sound speeds C and $\hat{C}$, and the speed of light c. The variable k appears in the products $K = kC$, $\hat{K} = k\hat{C}$, and $p = kc$. In most of the analysis p is treated as independent of K and $\hat{K}$. For example we may seek roots of the dispersion relation in which K (and hence $\hat{K}$) is prescribed to be real and small with respect to $|\Omega|$. Then we find s as a function of K while p is an independent physical parameter. On the other hand if we seek roots which are characteristic of electromagnetic propagation we find s as a function of p while K and $\hat{K}$ are treated as small quantities with respect to p by virtue of the fact that the plasma is non-relativistic.

At this point it is useful to sketch the connection between the lowest order results of R. L. Liboff[2] and those found by using the IFP model. Liboff used a generalization of the Krook model to construct a set of moment equations similar in kind to equations (3.16). He then constructed a dispersion relation and

investigated two limits near $k = 0$, that is, where $\tilde{k} = k$ times a suitable characteristic length is small. In the first limit he sought roots (in our notation) in which $s = s^{(o)} + s^{(1)}\tilde{k} + \ldots$. He found $s^{(o)}$ by dropping $\tilde{k}$ and higher order terms. In the second limit he sought those roots in which $s = s^{(1)}\tilde{k} + \ldots$. Again, he evaluated $s^{(1)}$ by consistently dropping higher order terms in his expansions. To the order of $s^{(o)}$ in the first limit and $s^{(1)}$ in the second limit the IFP model gives the same results as Liboff's. It is only to higher order that the two models differ. One may uncover all of Liboff's roots of the first kind ($s = s^{(o)} = \text{const.}$) from equations (3.23) by suitably expanding the integrals and dropping terms of the proper order. To get his roots of the second kind ($s = s^{(1)}\tilde{k}$) one uses, instead, equations (3.13), (3.16) and the equations for n and $\hat{n}$ which we have not written down since we do not need them in the sequel. Since both theories are equivalent to this order we do not give the details of these computations.

The next section contains the principal results of this investigation: the study of the Larmor resonances. It is with these resonances that the predictions of the two theories become different.

4. The Larmor Resonances

A. Introductory Remarks

The Larmor resonances are roots of the dispersion relation having the form

$$s = iN|\Omega| + R_N \tag{4.1}$$

where N is an integer and R_N is some function of k, θ, p, N and the physical constants. There is a corresponding set of resonances involving the ion Larmor frequency $\hat{\Omega}$. To be sure, this is a statement about dispersion relations rather than about physically measured resonances. For this reason the nature of the resonances depends on the collision model from which one derives the dispersion relation. Hopefully, measuring techniques will improve to the point where the predictions of competing models will be subject to experimental verification or rejection. Indeed, the predictions based on the IFP model differ significantly from those based on the Krook model and its modifications[1,2].

We need certain representative lengths for the rest of the discussion. These lengths, given by equations (4.1), are defined up to multiplicative constants which do not affect orders of magnitude.

$$\lambda = \text{the wavelength} = 1/k,$$

$$\lambda_c = \text{the collision mean free path} = C/\nu,$$

$$\lambda_\Omega = \text{the electron Larmor radius} = C/|\Omega|, \tag{4.2}$$

$$\hat{\lambda}_\Omega = \text{the ion Larmor radius} = \hat{C}/\hat{\Omega},$$

$$\lambda d = \text{the Debye distance} = (C^2/\omega_o^2)^{\frac{1}{2}} .$$

We do not distinguish between ion and electron mean free path since equations (2.10) and (2.14) indicate that they are of the same order.

The condition that $\underset{\sim}{k}$ be real is imposed throughout. In the region of $\underset{\sim}{k}$ space for which $(\lambda_\Omega/\lambda) << 1$, $(\hat{\lambda}_\Omega/\lambda) << 1$ and $(\lambda_c/\lambda) << 1$ we find two significant differences between the IFP and the Krook model Larmor resonances. The first is with respect to the angle θ between $\underset{\sim}{k}$ and $\underset{\sim}{B}_o$. From equation (3.23) one finds roots of the form (4.1) for arbitrary N at angles θ in the range

$$0 < \theta < \pi .$$

On the other hand the Krook models predict roots for arbitrary N only at $\theta = \pi/2$ and an extremely small angular cone about $\theta = \pi/2$ which decreases as $|N|$ increases. This point will be clarified in subsection B.

The second significant difference is in the multiplicity of roots of the form (4.1) for each given N. From the IFP dispersion relation we find an infinite number of roots for each N which differ, roughly, by integral multiples of ν: from the Krook

models we find but one. This reflects the fact that the IFP operator is unbounded and has a discrete spectrum[17] (zero and the negative integral multiples of ν), while the Krook operator is bounded and has but one eigenvalue $(-\nu)$.

One of the uses of a collision model is in the study of collisionless limits. By collisionless limit we mean a process by which the two body collision frequencies become negligible in some sense. The most interesting such process is non-uniform in that it retains the concept of collision mean free path. To reach it we require that $(\lambda_\Omega/\lambda) \ll 1$, $(\hat{\lambda}_\Omega/\lambda) \ll 1$, and keep $(\lambda_c/\lambda) = K/\nu$ fixed and much smaller than unity as ν becomes small. As we show in subsection C, for each integer N there will be a large number of Larmor resonances with negligible damping at arbitrary angles θ in the range $0 < \theta < \pi$. The degeneracy of such resonances about each $s = iN|\Omega|$ increases as ν becomes smaller. These results reflect the properties which distinguish the IFP from the Krook models. The latter models cannot give such results for the same reasons indicated above for the case where collision ~~less~~ are not negligible.

In subsection D we consider the uniform collisionless limit in the range $(\lambda_\Omega/\lambda) \ll 1$ and $(\hat{\lambda}_\Omega/\lambda) \ll 1$. We reach this limit by letting the collision frequencies vanish while all other quantities remain fixed. In the uniform limit both Krook and IFP models lead to dispersion relations which are equivalent to that derived by I.B. Bernstein[18] from the collisionless Boltzmann

equations. The only reason for elaborating on this case is that Bernstein's results for the Larmor resonances were given for longitudinal electric waves. That is, he applied the constraint that $\underset{\sim}{E}$ be parallel to $\underset{\sim}{k}$. This considerably simplifies the dispersion relation. R.M. Lewis and J.B. Keller[19], in their study of a Krook model dispersion relation, found that this constraint imposes certain conditions which must be satisfied by the transverse components of the dispersion tensor. The conditions are not satisfied, however, in the range $(\lambda_\Omega/\lambda) \ll 1$ at $\theta = \pi/2$. This is not to say that Bernstein's results are wrong in this range. Rather, one must resolve the question of whether or not the transverse electric waves are negligible by studying the analytic properties of the initial data. The dispersion equation itself does not give answers to such questions*. For this reason we present the results without the constraint to longitudinal waves. The principal effect of relaxing the constraint is that there are a threefold rather than a singlefold infinity of Larmor resonances for each species.

In subsection E we continue our study of Larmor resonances by considering a plasma with finite collision frequencies.

The first study is for θ in the range $0 < \theta < \pi$ and $(\lambda_\Omega/\lambda) \ll 1$, $(\hat{\lambda}_\Omega/\lambda) \ll 1$ and $(\lambda_c/\lambda) \ll 1$. At $\theta = \pi/2$ we compare the IFP results with those of R.L. Liboff[2]. We then

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* For this observation we are indebted to Prof. H. Weitzner.

study the fundamental Larmor resonances at $\theta = 0$ in the limit $(\lambda_c/\lambda) \ll 1$. Finally in subsection F we study the Larmor resonances in the range $(\lambda_c/\lambda) \gg 1$ at $\theta = 0$.

B. Analysis

Except for the case $(\lambda_c/\lambda) \gg 1$ the computations leading to the Larmor resonances rest on the following analysis.

We make the change of variables $y = z/|\Omega|$ in the integrals defined in appendix A. It is convenient to make all the frequencies in the matrices D_1 and D_2 (equations (3.21) and (3.22)) non dimensional with respect to $|\Omega|$. The dimensionless frequencies carry a bar $(\bar{\ })$: $\bar{s}$, $\bar{p}$, $\bar{\omega}_o$, $\bar{\nu}$, etc. The ion integrals and frequencies are treated identically with respect to $|\Omega|$ since we focus our attention on the electron resonances. There is a parallel development leading to the ion resonances.

The elements of D_1 and D_2 contain typical integrals of the form

$$I(d_1,d_2) = \int_0^\infty dz \exp\left\{ \bar{\phi} - (d_1\bar{\nu} + id_2)z \right\} , \tag{4.2}$$

where the integers d_1 and d_2 have the ranges

$$0 \leq d_1 \leq 4 \quad \text{and} \quad 0 \leq |d_2| \leq 2 ,$$

$$\bar{\phi} = \tau - (\bar{s} + \bar{\nu}\mu^2)z + \bar{\phi}_3 + \bar{\phi}_+ + \bar{\phi}_- , \tag{4.3}$$

$$\tau = \mu_3^2 - \mu_\perp^2 \cos\psi \ , \ \mu_3^2 = \frac{K^2\cos^2\theta}{\nu^2} \ , \ \mu_\perp^2 = \frac{K^2\sin^2\theta}{\nu^2+\Omega^2} \ , \ \mu^2 = \mu_3^2+\mu_\perp^2 \ ,$$

$$\bar{\phi}_3 = -\mu_3^2 \, e^{-\bar{\nu}z} \ , \qquad e^{\pm i\psi} = -\,(\nu\mp i\Omega)/(\nu\pm i\Omega) \ ,$$

$$\bar{\phi}_\pm = (\mu_\perp^2/2)\exp\left\{\pm\, i\psi - (\bar{\nu} \mp i)z\right\} \ .$$

The quantities $\bar{\phi}_3$, $\bar{\phi}_+$ and $\bar{\phi}_-$ are bounded functions of z in the range of integration. Therefore the expansion

$$e^{\bar{\phi}_3} = \sum_{\eta_3=0}^{\infty} \frac{\left(\bar{\phi}_3\right)^{\eta_3}}{\eta_3!}$$

converges absolutely as a function of z as do the similar expansions for $e^{\bar{\phi}_+}$ and $e^{\bar{\phi}_-}$. After substituting the expansions into (4.2) we integrate term by term for real part $(\bar{s}+\bar{\nu}\mu^2) > 0$. The result after a straightforward change of summation variables is

$$I(d_1 d_2) = \tag{4.4}$$

$$e^{\tau} \sum_{P=0}^{\infty} \sum_{N=-\infty}^{+\infty} \sum_{L=0}^{\infty} \left[\frac{(-\mu_3^2)^P (\mu_\perp^2/2)^{|N|+2L} \, e^{iN\psi}}{P!\,(|N|+L)!\,L! \left\{\bar{s}+\bar{\nu}(\mu^2+|N|+2L+P+d_1)-i(N-d_2)\right\}} \right] .$$

For $\underset{\sim}{k}$ (and hence μ^2) real the analytic continuation of $I(d_1,d_2)$ into the complex $\bar{s}$ plane is direct. The function $I(d_1,d_2)$ is analytic except at the poles in the denominators.

There is but a finite number of terms of the triple sum contributing to each pole. This number depends on the value of

2L + P. Since the poles are distinct, one can show that at an arbitrarily small distance $\varepsilon > 0$ from one of them the function~~s~~ $I(d_1,d_2)$ satisfies

$$|I(d_1,d_2)| < e^{\tau+\mu^2}/\varepsilon .$$

That is, $I(d_1,d_2)$ is bounded away from the poles.

There is an expansion similar to (4.4) for the ion integrals. The ion poles are distinct from the electron poles by virtue of the mass differences.

We use the expansions (4.4) to find the Larmor resonances as roots of the dispersion equation (3.23) for real $\underset{\sim}{k}$ in the parameter range:

$$(\lambda_c/\lambda) \ll 1, \ (\lambda_\Omega/\lambda) \ll 1 \ \text{ and } \ (\hat{\lambda}_\Omega/\lambda) \ll 1. \qquad (4.5)$$

In terms of the components of $\underset{\sim}{\mu}$ and $\hat{\underset{\sim}{\mu}}$ the inequalities (4.5) are equivalent to

$$|\mu_3| \ll 1, \ |\mu_\perp| \ll 1, \ |\hat{\mu}_3| \ll 1 \ \text{ and } \ |\hat{\mu}_\perp| \ll 1. \qquad (4.6)$$

The poles of expansion (4.4) suggest the possible Larmor resonances. For example, we fix $N = N_o > 1$ and guess that a possible root might be

$$\bar{s} + \bar{\nu}\mu^2 = \bar{s}_o + (1/\gamma)(\mu_\perp^2/2)^{N_o'}(-\mu_3^2)^{N_o''} , \qquad (4.7)$$

where

$$\bar{s}_o = iN_o - \bar{\nu}N_o,$$

γ and the numbers N_o' and N_o'' are to be determined. We look for the lowest order root in μ, a root for which γ is independent of μ and the magnitude of K and for which the sum $N_o' + N_o''$ is a minimum. Inspection of the integrals appearing in (3.23) shows that the lowest order terms containing γ come from $I(1,-1)$ and the product $\mu_\perp^2\, I(2,-2)$. We substitute (4.7) into $I(1,-1)$ and find that for $N_o' = N_o - 1$ and $N_o'' = 0$

$$I(1,-1) = \frac{1}{\bar{s}_o + \bar{\nu} - i} + \gamma\, \frac{e^{i(N_o-1)\psi}}{(N_o-1)!} + O(\mu_\perp^2) + O(-\mu_3^2)\ .$$

Similarly $\mu_\perp^2\, I(2,-2)$ contributes a term in γ plus higher order terms in $\mu_\perp^2$ and μ_3^2. All of the terms in (3.23) of order μ_3, $\mu_\perp$, $\hat{\mu}_3$, $\hat{\mu}_\perp$ and higher are dropped; thus, in the terms $\bar{\alpha}$, $\bar{\beta}$, $\bar{\eta}$ and in the corresponding ion terms $\bar{s}$ is replaced by $\bar{s}_o$.

In a similar way we find the other roots suggested by the poles of the integrals (4.4). For each such root the expansion of the determinant (3.23) leads to an equation which is either linear or quadratic in the unknown γ. We then find γ and verify the existence of the Larmor resonances as roots of the dispersion relation. The same procedure with appropriate modification leads to the ion resonances and to the various limiting cases to be discussed.

At this point we focus attention on the angular behavior of the Larmor resonances. We take as typical the resonance described by equation (4.7). The root may be written

$$\bar{s} + \bar{\nu}\mu^2 = iN_o - N_o\bar{\nu} + \gamma^{-1}\left\{\frac{\bar{K}^2}{\bar{\nu}^2+1}\ \frac{\sin^2\theta}{2}\right\}^{N_o-1}, \tag{4.8}$$

where $\bar{K}^2 = K^2/\Omega^2$ and $N_o > 1$.

The only restriction on angle is that θ lie in the open interval $0 < \theta < \pi$. The end points present the only non-uniformity since for $N_o > 1$ there is no resonance at $\theta = 0$ and $\theta = \pi$. Within the allowed angular range the form of the resonance changes continuously with angle. The Krook models, however, behave differently. To show this we study the Krook integral I' which corresponds to our integral $I(1,-1)$. The example comes from the paper of R.M. Lewis and J.B. Keller[19]. In our notation

$$I' = \int_0^\infty dz\ \exp(\phi' + iz) \tag{4.9}$$

where

$$\phi' = -\bar{K}^2 - (\bar{s}+\bar{\nu})z - \frac{\bar{K}_3^2 z^2}{2} + \frac{\bar{K}_\perp^2}{2}\left(e^{iz}+e^{-iz}\right),$$

$$\bar{K}_3^2 = \bar{K}^2\cos^2\theta\ ,\quad \bar{K}_\perp^2 = \bar{K}^2\sin^2\theta,\quad \bar{K}^2 = \bar{K}_3^2 + \bar{K}_\perp^2\ .$$

The functions $\frac{\bar{K}_\perp^2}{2}\, e^{\pm iz}$ are bounded in the region of integration so that the expansions of the exponentials converge.

After some manipulation

$$I' = e^{-\overline{K}^2} \sum_{N=-\infty}^{+\infty} \sum_{L=0}^{\infty} \frac{(\overline{K}_\perp^2/2)^{|N|+2L}}{(|N|+L)!D!} \int_0^\infty dz \exp\left[-(\overline{s}+\overline{\nu}-i(N+1))z - \frac{\overline{K}_3^2 z^2}{2}\right] . \tag{4.10}$$

We seek the resonance characterized by $N_o > 1$. To lowest order in $K_\perp^2$, the two contributing terms are $N = L = 0$, and $N = N_o-1$ and $L = 0$. The latter term is designated I'_{N_o}.

$$I'_{N_o} = \frac{(\overline{K}_\perp^2/2)^{N_o-1}}{(N_o-1)!} \int_0^\infty dz \exp\left[-(\overline{s}+\overline{\nu}-iN_o)z - \frac{\overline{K}_3^2 z^2}{2}\right] . \tag{4.11}$$

We require that $0 < \overline{K}_3^2 << 1$. Let

$$\tau = |\overline{K}_3| z \quad \text{and} \quad \sigma = \frac{(\overline{s}+\overline{\nu}-iN_o)}{|\overline{K}_3|} , \tag{4.12}$$

then

$$I'_{N_o} = \frac{(\overline{K}_\perp^2/2)^{N_o-1}}{(N_o-1)!|\overline{K}_3|} I_{(\sigma)} \tag{4.13}$$

where

$$I_{(\sigma)} = \int_0^\infty d\tau \exp(-\sigma\tau - \tau^2/2) . \tag{4.14}$$

One may get an asymptotic expansion of $I_{(\sigma)}$ for $|\sigma| >> 1$ by integrating by parts several times, and a Taylor expansion for $|\sigma| << 1$ by expanding $e^{-\sigma\tau}$ and integrating the required number of terms. These expansions in slightly different form are found

in reference [19].

For $|\sigma| \gg 1$

$$I_{(\sigma)} \sim \frac{1}{\sigma}\left(1 - \frac{1}{\sigma^2} + O\left(\frac{1}{\sigma^4}\right)\right) . \tag{4.15}$$

For $|\sigma| \ll 1$

$$I_{(\sigma)} = \sqrt{\frac{\pi}{2}} - \sigma + O(\sigma^2) . \tag{4.16}$$

We suppose that $|\sigma| \gg 1$ and substitute the first term of (4.15) into (4.14). The result is

$$I'_{N_o} = \frac{(\overline{K}_\perp^2/2)^{N_o-1}}{(N_o-1)!}\left(\frac{1}{\overline{s}+\overline{\nu}-iN_o}\right) . \tag{4.17}$$

We now guess that

$$\overline{s} = -\overline{\nu} + iN_o + (1/\gamma')(\overline{K}_\perp^2/2)^{N_o-1} . \tag{4.18}$$

To lowest order in the small quantities $\overline{K}_\perp$ and $\overline{K}_3$,

$$I' = \frac{1}{i(N_o-1)} + \frac{\gamma'}{(N_o-1)!} . \tag{4.19}$$

The determination of γ' from the appropriate Krook dispersion formula then follows immediately. The procedure is valid provided that $|\sigma| \gg 1$ or equivalently that

$$\left|\frac{1}{\gamma'}\right|\left(\frac{K_\perp^2}{2}\right)^{N_o-1}\left(\frac{1}{|\overline{K}_3|}\right) \gg 1.$$

If we suppose that $|\gamma'|$ is $O(1)$ the inequality becomes

$$|\cot\theta| \ll \left|\frac{\overline{K}_\perp}{2}\right| \left(\frac{\overline{K}_\perp^2}{2}\right)^{N_o-2} \quad . \tag{4.20}$$

Now suppose that $|\sigma| \ll 1$ and that we still wish to find a root of the form (4.18). Substitution of (4.16) and (4.18) into (4.13) shows that this is impossible. For example let $|\sigma| \leq \left|\frac{\overline{K}_\perp}{2}\right| \ll 1$ then for

$$|\cot\theta| \geq (\overline{K}_\perp^2/2)^{N_o-2} \tag{4.21}$$

there can be no root of the form (4.18). The region of validity might be expanded for particular values of the physical constants and N_o all of which determine γ'. None the less the conclusion remains the same: there will be an angular region about $\pi/2$ beyond which there can be no resonances of the form (4.18). One notes from equation (4.21) that the angular region shrinks with increasing N_o. No such restrictions appear in the Larmor resonances deduced from equation (3.23).

I.B. Bernstein[18] observed a similar angular restriction on the Larmor resonances in the collisionless case. More precisely, setting $\overline{\nu} = 0$ in equations (4.9) to (4.19) gives roughly the same angular region of existence for the Larmor resonances for both the collisionless and the collisional Krook equations. Again, the region might be expanded or contracted for special values of the constants which determine γ'.

In general there are no roots of the form $\bar{s} = \pm i - \bar{\nu}$ to lowest order. Rather one obtains from the dispersion relation (3.23) the same plasma electromagnetic roots found, for example, in R.L. Liboff's study[2]. His paper contains a detailed study of these roots and of the special limiting cases leading to the appropriate Larmor resonances which bear the indices $N = \pm 1$. We do not, therefore, treat these limiting cases here. There are, however, roots of the form $\bar{s} + \bar{\nu}\mu^2 = \pm i - L\bar{\nu} + R_L$, where $L = 2,3,4\ldots$, and R_L is a function of $\mu_\perp^2$, μ_3^2, θ, $\bar{\omega}_o^2$, $\bar{p}^2$, L, the frequencies and the masses. Such roots can not be found from dispersion relations derived from the Krook models.

Certain comments are in order before we proceed further. By seeking the lowest order roots we lose three other possible roots at each resonance since each pole appears linearly in each element of the matrices D_1 and D_2 of equation (3.21) and (3.22). To obtain the higher order roots one would need to keep enough orders of μ_3^2 and $\mu_\perp^2$ so that the expansions of the elements of equation (3.23) would be consistent. Evaluation of the dispersion equation would be equivalent to solving a quartic equation in the unknown function γ which now depends on the components of $\underset{\sim}{\mu}$.

In order to simplify the display of certain formulae involving the electron resonances we consider the limit $\tilde{m} \rightarrow \infty$. In this limit the ions become a uniform neutralizing background, the collision frequencies $\hat{\nu}_1$ and ν_2 vanish, and the dispersion equation (3.23) reduces to

$$\det D_1 = 0 . \tag{4.22}$$

The conclusions drawn from (4.22) relevant to the Larmor resonances are also valid for $\hat{m}$ finite. The only difference appears in the γ's which for $\hat{m}$ finite contain additional terms involving multiples of the elements of $\hat{D}_1$ and $\hat{D}_2$ and the $\bar{\alpha}$ and $\bar{\eta}$ terms again contain the appropriate ion contributions. The elements of $\hat{D}_1$ and $\hat{D}_2$ are finite since the poles in the ion integrals are separate from the poles in the electron integrals. (This conclusion is a consequence of the analysis following equation (4.41).) Since we can translate the results for ion mass infinite we can also translate statements about electron Larmor resonances into statements about ion Larmor resonances.

C. <u>The Non-Uniform Collisionless Limit</u>

We study the non-uniform limit by first choosing the small parameter $\varepsilon << 1$ and then specifying

$$\bar{\nu} = O(\varepsilon)$$

and (4.23)

$$\bar{K} = O(\varepsilon^2)$$

so that $(\lambda_c/\lambda) = (\bar{K}/\bar{\nu}) = O(\varepsilon)$.

The choice $\bar{K} = O(\varepsilon^{1+\Delta})$ where $\Delta > 0$ would do as well. It is easier to follow the orders of magnitude consistently with (4.23). We illustrate the computations in the limit $\hat{m} \longrightarrow \infty$ and

assert that the conclusions are valid for finite $\hat{m}$ by virtue of the discussion at the end of section 4B.

A consequence of equations (4.23) is that $|\mu_\perp|$ is at most of order ε^2 and $|\mu_3|$ is at most of order ε. One may use the expansions (4.4) in the dispersion equation (4.22). A natural subdivision in this study is given by the angular domains defined by $(\sin\theta \cos\theta) \neq 0$, $\cos\theta = 0$, and $\sin\theta = 0$.

(i) <u>$\sin\theta \cos\theta \neq 0$</u>

We look first for the root about

$$\bar{s}_o = iN_o - \bar{\nu}N_o \ , \quad N_o > 1.$$

Our previous guess (equation 4.8) takes the form

$$\bar{s} + \bar{\nu}\mu^2 - \bar{s}_o = \frac{(\mu_\perp^2/2)^{N_o-1} e^{i(N_o-1)\psi} N_o(\bar{s}_o+\bar{\nu}-i)}{(N_o-1)!\ \gamma} \ , \tag{4.24}$$

where γ is the unknown quantity. We substitute (4.24) into the integrals of D_1 given by (3.21) and drop all terms of order ε and higher. To lowest order

$$\begin{aligned} I_{33} &= H_3 = 1/iN_o \ , \\ I_{--} &= H_- = 1/i(N_o+1) \ , \\ I_{++} &= H_+ = (1+\gamma)/i(N_o-1) \ , \\ I_{+-} &= I_{3+} = I_{3-} = I_{3T} = I_{+T} = I_{-T} = I_{TT} = 0. \end{aligned} \tag{4.25}$$

Also to lowest order

$$\begin{aligned} \bar{\alpha} &= -\bar{\omega}_o^2\ \bar{s}_o/(\bar{s}_o^2 + \bar{p}^2) \ , \\ \bar{\eta} &= \bar{\omega}_o^2\bar{p}^2/\bar{s}_o(\bar{s}_o^2 + \bar{p}^2) \ . \end{aligned} \tag{4.26}$$

If in addition $(N_o\varepsilon) << 1$ is imposed and we drop terms of order $(N_o\varepsilon)$ and higher, then $\bar{s}_o$ may be replaced with iN_o unless $\bar{p}^2 - N_o^2 = 0$ for some particular values of $\bar{p}$ and N_o. After we substitute (4.25) and (4.26) with $\bar{s}_o = iN_o$ into (4.22) we need only solve an equation linear in γ, the unknown quantity. Therefore γ is known to lowest order in terms of θ, N_o, $\bar{\alpha}$ and $\bar{\eta}$. Should γ vanish for exceptional values of these parameters one must compute γ to higher order in ε. The evaluation of γ establishes that the Larmor resonance

$$\bar{s} = iN_o - N_o\bar{\nu} + O(\varepsilon^3) \tag{4.27}$$

is a root of the dispersion equation (4.22). In a similar way one can demonstrate by the same procedure that there are either one or two lowest order roots for each $L \geq 1$ of the form

$$\bar{s} = iN_o - \bar{\nu}(|N_o| + L) + O(\varepsilon^3) \tag{4.28}$$

where

$$|N_o| > 1$$

and

$$L = 1,2,\ldots \quad .$$

The number is two if $O(\mu_3^2) = O(\varepsilon^4) = O(\mu_\perp^2)$ and one if $O(\mu_3^2) = O(\varepsilon^2)$, $O(\mu_\perp^2) = O(\varepsilon^4)$. The terms $O(\varepsilon^3)$ in both (4.27) and (4.28) are contributed by $\bar{\nu}\mu^2$. The terms involving γ are generally of higher order although, as mentioned above, peculiar combinations of $\bar{p}^2$, $\bar{\omega}_o^2$, N_o, and θ might make $\gamma = O(\varepsilon^2)$. We

assume that in general such is not the case. One may go to the limit $\varepsilon = 0$ in the following way. For the root (4.27), for example, we substitute equation (4.24) into the integrals of (4.22) and then take the limit $\varepsilon = 0$ and hence $\bar{\nu} = 0$ and $\bar{K} = 0$ by (4.23). The elements of the dispersion equation for γ remain finite and we say that in the collisionless limit the dispersion equation is compatible with the Larmor resonance $\bar{s} = iN_o$. If we fix N_o and substitute into (4.22) successively for each value of L in (4.28) the form corresponding to (4.24) and go to the limit $\varepsilon = 0$, then $\bar{s} = iN_o$ is again a resonance compatible with the dispersion formula (4.22). In this sense the Larmor resonance $\bar{s} = iN_o$, for $|N_o| > 1$, is infinitely degenerate. For $N_o = \pm 1$ the limit $\varepsilon = 0$ does not exist except when $\bar{\omega}_o^2/\bar{\nu}$ remains finite as $\bar{\nu}$ goes to zero. We see this by a closer inspection of the term I_{++}. For example suppose we look at the resonance about $\bar{s} + \bar{\nu}\mu^2 = i-2\bar{\nu}$. To lowest order

$$I_{++} = (\bar{s}+\bar{\nu}\mu^2-i+\bar{\nu})^{-1} + (-\mu_3^2)(\bar{s}+\bar{\nu}\mu^2-i+2\bar{\nu})^{-1} \quad . \tag{4.29}$$

If we suppose

$$\bar{s} + \bar{\nu}u^2 - i + 2\bar{\nu} = (-\mu_3^2)/\gamma \tag{4.30}$$

then to lowest order

$$I_{++} = -\frac{1}{\bar{\nu}} + \gamma \ . \tag{4.31}$$

From (4.31) $\gamma = \bar{\nu}^{-1}$ and as a next approximation one might write $\gamma = (1+\bar{\nu}A)/\bar{\nu}$, substitute into the dispersion relation (4.22), and

find that H_+ and the term I_{3+} give with I_{++} a quadratic expression for A. If $\bar{\omega}_o^2$ remains finite as ε and hence $\bar{\nu}$ vanish, the terms in I_{++} blow up. However if $\bar{\omega}_o^2/\bar{\nu}$ remains finite as $\bar{\nu}$ goes to zero, and if we postulate that $(\bar{\omega}_o^2\gamma)$ remains finite, we conclude that the Larmor resonance $\bar{s} = i$ is consistent with the dispersion equation.

In general for $0 < \varepsilon << 1$ one can conclude that there are resonances of the form

$$\bar{s} = \pm i - \bar{\nu}(1+L) + O(\varepsilon^3) , \tag{4.32}$$

where $L = 1,2,3 \ldots$

but that the limit $\varepsilon = 0$ does not exist in the sense described above for $|N_o| > 1$.

(ii.) <u>$\cos\theta = 0$</u>

In this case $\mu_3 = 0$ and $\mu_\perp = \mu$ in the integrals of Appendix A. Assuming that no combinations of $\bar{\omega}_o^2$, $\bar{p}^2$ and N_o will cause the lowest order determination of the γ's to vanish we may write the dispersion equation (4.22) in the form

$$(-1+\bar{\alpha}\, I_{33}) \det D_1' = 0 \tag{4.33}$$

where

$$\det D_1' = \begin{vmatrix} (-1+(\bar{\alpha} - \bar{\eta}/2)H_+) & (-\bar{\eta}/2\, H_+) \\ (-\bar{\eta}/2\, H_-) & (-1+(\bar{\alpha} - \bar{\eta}/2)H_-) \end{vmatrix} . \tag{4.34}$$

In the exceptional cases one might have to restore $\bar{\alpha}\, I_{+-}$

to (4.34) and include the row and column containing I_{+T} , I_{-T} , H_T and I_{TT} . We suppose that the exceptional cases do not arise.

Of the lowest order roots (4.28) those for L odd no longer appear at $\cos\theta = 0$ when we follow the usual procedure for finding the roots of (4.34). However, at angles arbitrarily close to $\pi/2$ equation (4.22) does give roots for odd L. Such apparent non-uniformities are a result of working only with the dispersion equation and ignoring the initial data. A more obvious example occurs at $\sin\theta = 0$.

The term $(-1+\bar{\alpha}\, I_{33})$ in equation (4.33) does contribute roots for odd L. When $\mu_3^2 = O(\varepsilon^2)$ these were higher order roots and did not appear in the $\cos\theta \cdot \sin\theta \neq 0$ case. When $O(\mu_3^2) = O(\mu_\perp^2) = O(\varepsilon^4)$ these were half of the two sets of roots found for each $L \geq 1$. Equations (4.27) and (4.28) remain the same except that $O(\varepsilon^3)$ becomes $O(\varepsilon^5)$ with the exception of the combination $|N_o| = 2$ and $L = 0$ for which case the lowest order in ε is $O(\varepsilon^4)$. The limit $\varepsilon = 0$ does not change. Also the resonances (4.32) remain the same as does the conclusion that the limit $\varepsilon = 0$ does not exist in general.

(iii) <u>$\sin\theta = 0$</u>

Here $\mu_\perp = 0$ and $\mu_3 = \mu$ in the integrals of appendix A. The only terms contributing a Larmor resonance are

$$(-1 + \bar{\alpha}\, I_{++})\,(-1 + \bar{\alpha}\, I_{--}) = 0. \tag{4.35}$$

For $N_o = \pm 1$ the integral $I_{\pm\pm} = \int_0^\infty e^{\bar{\phi}-(\bar{\nu}\pm i)z} dz$ contributes.

The roots are given by

$$\bar{s} + \bar{\nu}\mu^2 = \bar{s}_o + \left(\frac{-\mu^2}{L!}\right)^L \gamma^{-1} \tag{4.36}$$

where

$$L = 1,2,3, \ldots ,$$

$$\bar{s}_o = \pm i - \bar{\nu}(L+1) ,$$

$$\gamma = \frac{1}{L\bar{\nu}} - \frac{(\bar{s}_o^2+\bar{p}^2)}{\bar{\omega}_o^2 \, \bar{s}_o} .$$

We see again that $\varepsilon = 0$ is an improper limit unless $\bar{\omega}_o^2/\bar{\nu}$ remains finite as $\bar{\nu} = O(\varepsilon)$ goes to zero and $\bar{\omega}_o^2\gamma$ also remains finite. The condition that $\bar{\omega}_o^2/\bar{\nu}$ be finite as $\bar{\nu}$ goes to zero can be achieved by letting the electron density n_o go to zero since both $\bar{\omega}_o^2$ and $\bar{\nu}$ are proportional to n_o. The are no resonances for $|N_o| \geq 2$ at $\sin\theta = 0$. At angles arbitrarily close to 0 or π we get all of the resonances (4.28). As before, such non uniform behavior is due to the fact that we omit the initial conditions and deal with the dispersion equation alone.

D. <u>The Uniform Collisionless Limit at $\theta = \pi/2$</u>

Again we illustrate with the dispersion relation (4.22) for ion mass infinite. To reach the uniform collisionless limit we

let $\bar{\nu}$ go to zero and keep all other quantities fixed. At $\theta = \pi/2$ the dispersion equation (4.22) becomes

$$\det D_1'(0) = \begin{vmatrix} [1+\bar{\alpha}\, I_{33}] & 0 & 0 \\ 0 & [-1+\bar{\alpha}\, I_{++} - \frac{\bar{\eta}}{2} H_+] & \frac{1}{2}[\alpha I_{+-} - \bar{\eta} H_+] \\ 0 & \frac{1}{2}[\bar{\alpha} I_{+-} - \bar{\eta} H_-] & [-1+\bar{\alpha} I_{--} - \frac{\bar{\eta}}{2} H_-] \end{vmatrix} = 0, \tag{4.37}$$

where

$$I_{33} = \int_0^\infty e^{\bar{\phi}} dz,$$

$$I_{++} = \int_0^\infty e^{\bar{\phi}} \left\{ e^{+iz} + \left(\bar{K}^2/2\right)(1-e^{+iz})^2 \right\} dz,$$

$$I_{--} = \int_0^\infty e^{\bar{\phi}} \left\{ e^{-iz} + \left(\bar{K}^2/2\right)(1-e^{-iz})^2 \right\} dz, \tag{4.38}$$

$$I_{+-} = -\bar{K}^2 \int_0^\infty e^{\bar{\phi}} (1-e^{+iz})(1-e^{-iz}) dz,$$

$$H_{\pm} = \pm\, i\bar{s} \int e^{\bar{\phi}} (1-e^{\pm iz}) dz,$$

$$\bar{\phi} = -K^2 - \bar{s}z + \bar{K}^2/2\, (e^{+iz} + e^{-iz}) .$$

Typical integrals have the form

$$I(d) = \int e^{\bar{\phi} + idz} \tag{4.39}$$

for d an integer in the range $-2 \leq d \leq 2$.

We may expand the exponential of the bounded functions $(\overline{K}^2/2)e^{\pm iz}$ and integrate for real part $\overline{s} > 0$. The result is

$$I(d) = e^{-\overline{K}^2} \sum_{N=-\infty}^{+\infty} \sum_{L=0}^{\infty} \frac{(\overline{K}^2/2)^{|N|+2L}}{(|N|+L)!L!(\overline{s}-i(N+d))} \quad . \tag{4.40}$$

To effect the analytic continuation we note that $I(d)$ is bounded away from the well separated infinite set of poles. If $\overline{s}$ is a distance $0 < \delta < \frac{1}{2}$ from one of them then

$$|I(d)| < 1/\delta \ .$$

Consequently, since the ion poles and the electron poles are distinct, one may apply the results for $\hat{m}$ infinite to $\hat{m}$ finite as before.

We suppose that $\overline{K}(= K/|\Omega|) << 1$ and apply the procedures of 4B to (4.37) and (4.38). The integrals $I_{++} = H_+$, $I_{--} = H_-$ contribute the lowest order resonances.

$$\overline{s} = iN\left(1 + \frac{\overline{\omega}_o^2}{|N|!}(\overline{K}^2/2)^{|N|-1} \dot{R}_{|N|}\right) \tag{4.41}$$

where $|N| > 1$ and

$$R_{|N|} = \left\{ \frac{(|N|\overline{\omega}_o^2/(1+|N|)) + (\overline{p}^2/2) - N^2}{\overline{p}^2\left(1-\overline{\omega}_o^2/(N^2-1)\right) - N^2\left(1-\overline{\omega}_o^2/(N^2-1)\right)^2 + \overline{\omega}_o^4/(N^2-1)^2} \right\}. \tag{4.42}$$

There are in principle three roots obtained by including higher order terms in $\overline{K}^2$. According to (4.41) there are an infinite number of Larmor resonances at $\theta = \pi/2$.

Roughly speaking, as soon as

$$|\cot\theta| \geq \frac{R_{|N|}\overline{\omega}_o^2}{|N|!}\left(\frac{\overline{K}^2 \sin^2\theta}{2}\right)^{|N|-2} \tag{4.43}$$

the resonance (4.41) is no longer a solution of the dispersion equation. This follows from the statements subsequent to equation (4.21) with the refinement afforded by the substitution of

$$(1/\gamma') = \frac{\overline{\omega}_o^2 R_{|N|}}{|N|!} \quad .$$

We observe that the formal limit $|\overline{p}| \rightarrow \infty$ in (4.42) gives the results one can compute from I.B. Bernstein's longitudinal electric wave dispersion relation[18].

E. Larmor Resonances with Collisions

We choose the small parameter $\varepsilon << 1$ and require that

$$\overline{K} = (\lambda_\Omega/\lambda) = O(\varepsilon), \tag{4.44}$$

$$\overline{\nu} = O(1).$$

This choice insures that

$$\begin{aligned} &(\lambda_c/\lambda) = O(\varepsilon), \\ &|\mu_3| \text{ is } O(\varepsilon) \text{ or higher} \\ &|\mu_\perp| \text{ is } O(\varepsilon) \text{ or higher .} \end{aligned} \tag{4.45}$$

The order of $|\mu_3|$ depends on $\cos\theta$ and that of $|\mu_\perp|$ on $\sin\theta$.

Following the procedures of section 4B with respect to the dispersion formula (4.22) for $\hat{m}$ infinite we compute the Larmor resonances in the ranges: $(\sin\theta\cos\theta) \neq 0$, $\cos\theta = 0$, and $\sin\theta = 0$.

(i) <u>$(\sin\theta\cos\theta) \neq 0$</u>

The formulas in this range are given for both $|\mu_3|$ and $|\mu_\perp|$ of order ε. There are in principle four roots for each resonance. As one approaches $\cos\theta = 0$ or $\sin\theta = 0$ the lowest order root might switch from one of the four to another. The order of ~~magninutde~~ magnitude of the term multiplying the γ's might change but the existence of the roots will not. In the mid range $|\mu_3| = O(\varepsilon)$ and $|\mu_\perp| = O(\varepsilon)$ we get the Larmor resonances

$$\bar{s} = iN - \bar{\nu}(\mu^2+|N|+L) + O(\bar{K}^{2(|N|+L-1)}) \tag{4.46}$$

for

$N \neq 0$, $L = 0$ or 1, and $(|N|+L) > 1$;

$$\bar{s} = iN - \bar{\nu}(\mu^2+|N|+L) + O(\bar{K}^{2(|N|+L-2)}) \tag{4.47}$$

for

$N \neq 0$, $L = 2,3, \ldots$.

There are roots for $N = 0$ and $L = 2,3, \ldots$, but they are not Larmor resonances. The resonances (4.47) are contributed by the integral I_{TT} and represent therefore the influence of the

the temperature terms. The resonances (4.46) come from those terms involving $\underset{\sim}{E}$ and the macroscopic velocity $\underset{\sim}{u}$. If one could neglect temperature by arranging the initial conditions in some suitable manner, then the resonances (4.47) would come from the terms involving $\underset{\sim}{E}$ and $\underset{\sim}{u}$ and the exponents of the order of $\overline{K}$ would be $2(|N|+L-1)$ as in (4.46).

(ii) $\underline{\cos\theta = 0}$.

Here $\mu_3 = 0$ and $\mu_\perp^2 = \overline{K}^2/(1+\overline{\nu}^2)$. At this point one can compare the Larmor resonances computed from R.L. Liboff's[2] Krook model with those computed from equation (4.22). From (4.22) one gets

$$\overline{s} = iN - \overline{\nu}(\mu_\perp^2+|N|) + O(\mu_\perp^{2(|N|-1)}) \tag{4.48}$$

for $|N| > 1$. From Liboff's model one gets

$$\overline{s} = iN - \overline{\nu} + O(\overline{K}^{2(|N|-1)}) \tag{4.49}$$

for $|N| > 1$. As $|N|$ gets larger the damping of the IFP resonances increases while the Krook resonances have the same damping factor. In addition to (4.48) we have for each N the infinite set of resonances from (4.22)

$$\overline{s} = iN - \overline{\nu}(\mu_\perp^2+|N|+L) + O(\mu_\perp^2)^{\Delta L}) \tag{4.50}$$

where

$\Delta_L = |N| + L - 1$ for L = odd integer,

$\Delta_L = |N| + L - 2$ for L = even integer > 0 .

Equation (4.50) also gives Larmor resonances for $N = \pm 1$. These do not appear in the Krook theories.

One observes that in the uniform limit $\bar{\nu}$ goes to zero both equations (4.48) and (4.49) lead to the resonances (4.41). The resonances (4.50) do not exist in the uniform collisionless limit so that no degeneracy appears.

The infinite set of Krook resonances (4.49) exist only at $\cos\theta = 0$. As one departs from $\theta = \pm \pi/2$ the resonances begin to disappear roughly at the angle given by (4.21). This of course is markedly different from the behavior of the IFP resonances.

(iii) $\underline{\sin\theta = 0}$

Here $\mu_\perp = 0$ and $|\mu_3| = (\bar{K}/\bar{\nu}) = (\lambda_c/\lambda) = O(\varepsilon)$.

The Larmor resonances come from the two uncoupled factors of (4.22)

$$\begin{aligned} -1 + \bar{\alpha} I_{++} &= 0, \\ -1 + \bar{\alpha} I_{--} &= 0 . \end{aligned} \tag{4.51}$$

The results are

$$\bar{s} = \pm i - \bar{\nu}(\mu_3^2 + 1 + L) + O(\mu_3^{2L}) \tag{4.52}$$

for

$L = 1,2,3 \ldots$.

These roots do not appear in the Krook model theories.

F. <u>Larmor resonances at</u> $\theta = 0$ <u>for</u> $(\lambda_c/\lambda) \gg 1$.

Here $\mu_\perp = 0$ and $\mu = \mu_3 = (\overline{K}/\overline{\nu}) = (\lambda_c/\lambda) \gg 1$. At $\theta = 0$ equations (4.51) involve the Larmor resonances. The first of equations (4.51) is

$$1 = \left(\overline{\nu} - \frac{\overline{\omega}_o^2\overline{s}}{\overline{s}^2+\overline{p}^2}\right)\int_o^\infty \exp[-(\overline{s}+\overline{\nu}(1+\mu^2)-i)z + \mu^2(1-e^{-\overline{\nu}z})]dz \ . \qquad (4.53)$$

We prescribe μ to be real and positive and require that $\overline{\nu}$ be non vanishing although it may be small. The integral in equation (4.53) is related to the incomplete gamma function*. To show this one changes variables according to the scheme

$$\begin{aligned} t &= \mu^2 e^{-\overline{\nu}y} \\ \sigma &= (\overline{s}-i+\overline{\nu}(1+\mu^2))/\overline{\nu} \\ \tilde{\alpha} &= \overline{\alpha}/\overline{\nu} = (1 - \overline{\omega}_o^2\overline{s}/\overline{\nu}(\overline{s}^2+\overline{p}^2)) \end{aligned} \qquad (4.54)$$

Equation (4.53) transforms to

$$1 = \tilde{\alpha}\left\{e^{\mu^2}\mu^{-2\sigma}\int_o^{\mu^2} dt.t^{(\sigma-1)}e^{-t}\right\}$$

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* We are indebted to Prof. H. Weitzner for this observation.

or

$$1 = \tilde{\alpha} \left\{ e^{\mu^2} \mu^{-2\sigma} \Gamma(\sigma) - \Phi(\sigma,\mu^2) \right\} \tag{4.55}$$

where*

$$\Gamma(\sigma) = \int_0^\infty t^{(\sigma-1)} e^{-t} \, dt, \tag{4.56}$$

$$\Phi(\sigma,\mu^2) = e^{\mu^2} \mu^{-2\sigma} \int_{\mu^2}^\infty t^{(\sigma-1)} e^{-t} \, dt. \tag{4.57}$$

The integrals (4.56) and (4.57) converge for real part of $\sigma > 0$. We will look for roots of (4.55) for real part of $\sigma < 0$. The analytical continuation of $\Gamma(\sigma)$ from real part $\sigma > 0$ to real part $\sigma < 0$ is effected by the identity

$$\Gamma(\sigma) = \frac{\pi}{\Gamma(1-\sigma)\sin \pi\sigma} . \tag{4.58}$$

It is convenient to introduce the change of variables

$$Z = 1-\sigma \tag{4.59}$$

and to consider real part $Z > 0$ as the domain of interest. With (4.59) and the variable change $t' = (t/\mu^2 - 1)$ the integral (4.57) transforms to

$$\Phi(Z,\mu^2) = \int_0^\infty e^{-t'\mu^2} (1+t')^{-Z} \, dt' . \tag{4.60}$$

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* All of the formulas and expansions relating to the Gamma function are taken from Chapter XII of reference [20].

Not only is $\bar{\Phi}$ convergent for real part $Z > 0$ but one can also bound it from above and below. From above one sets $t' = 0$ in the exponential and gets

$$| \bar{\Phi} | < \frac{1}{Z_r - 1} \quad \text{where} \quad Z_r = \text{real part} \quad Z > 0 \ , \tag{4.61}$$

or one sets $t' = 0$ in $(1+t')$ and gets

$$| \bar{\Phi} | < \frac{1}{\mu^2} \ . \tag{4.62}$$

To bound from below one uses the inequality $(1+t') < e^{t'}$ and gets the bound

$$\frac{1}{Z_r + \mu^2} < | \bar{\Phi} | \ . \tag{4.63}$$

The dispersion equation (4.55) is now

$$1 = \tilde{\alpha} \left\{ \frac{e^{\mu^2} \mu^{2(Z-1)} \pi}{\Gamma(z) \sin \pi Z} - \bar{\Phi} \right\} \ . \tag{4.64}$$

The roots of (4.64) are found by a process of successive approximations. (i) We need the first approximation of Z to be large enough to cancel the large quantity $(e^{\mu^2} \mu^{2(Z-1)})$. (ii) With this first approximation, labeled Z_1, we use (4.59) and (4.54) to evaluate $\tilde{\alpha}$, and estimate the bounds of $\bar{\Phi}$. (iii) The next approximation is gotten by looking for the root $Z = (Z_1 + h)$, where h is bounded and independent of μ, to satisfy the rest

of the dispersion formula (4.64).

(i) Since we seek those roots for which the real part of Z is large and positive one is justified in replacing $\Gamma(Z)$ in (4.64) with its asymptotic expansion

$$\Gamma(Z) \sim (2\pi)^{\frac{1}{2}} Z^{(Z-\frac{1}{2})} e^{-Z} (1+O(\tfrac{1}{Z}) + \ldots) . \tag{4.65}$$

Neglecting the higher order terms we get

$$1 = \tilde{\alpha} \left\{ \frac{\exp[\mu^2+Z-\log \mu - \log (Z/\mu^2)\cdot(Z-\frac{1}{2})]}{\sin \pi Z\cdot(2/\pi)^{\frac{1}{2}}} - \Phi \right\} . \tag{4.66}$$

We guess that

$$Z_1 = d_o\mu^2 + a_o \log \mu . \tag{4.67}$$

Substituting (4.67) into the highest order terms in the exponential gives

$$(1+d_o-d_o\log d_o)\mu^2 - (1+a_o\log d_o) \log \mu = 0 .$$

Thus d_o and a_o are independent of μ for μ large and are given by

$$\begin{aligned} d_o &= 3.591, \\ a_o &= -1/\log d_o = -.782. \end{aligned} \tag{4.68}$$

Therefore Z_1 is known precisely.

(ii) The first approximation to $\bar{s}$ is

$$\bar{s}_1 = i - \bar{\nu}(1+d_o)\mu^2, \tag{4.69}$$

where the log μ term has been dropped since it is dominated by the μ^2 terms.

The first approximation to $\tilde{\alpha}$ is denoted by $\tilde{\alpha}_{(1)}$. We suppose the μ^2 is sufficiently large so that $|\tilde{\alpha}_{(1)}|$ is $O(1)$ and write

$$\tilde{\alpha}_{(1)} = |\tilde{\alpha}_{(1)}| e^{i\rho_o} . \tag{4.70}$$

(iii) The term $\bar{\Phi}$ is bounded in absolute magnitude by $1/\mu^2$ so that it may be dropped when $|\tilde{\alpha}_{(1)}|$ is $O(1)$.

Since Z_1 is known we write

$$Z_1 = 2N_1 + \Delta \tag{4.71}$$

where

N_1 is a positive integer and $0 \leq \Delta < 2$.

The next approximation is

$$Z = Z_1 + h \tag{4.72}$$

where h is to be small compared with log μ. We substitute (4.70), (4.71) and (4.72) into (4.66) and write $\sqrt{\frac{\pi}{2}}\,|\tilde{\alpha}_{(1)}|$ as e^{δ} and

$$h = x + iy \quad . \tag{4.73}$$

Then (4.66) becomes

$$\sin \pi(x+\Delta+iy) = e^{(\delta-(x-\frac{1}{2})\log d_o)}\, e^{i(\rho_o - y \log d_o)} .$$

We equate real parts and imaginary parts.

$$\sin \pi(x+\Delta)e^{-[\delta-(x-\frac{1}{2})\log d_o]} = \sin(y\log d_o+\rho') \cosh \pi y,$$

$$\cos \pi(x+\Delta)e^{-[\delta-(x-\frac{1}{2})\log d_o]} = \cos(y\log d_o+\rho') \sinh \pi y, \tag{4.74}$$

where

$$\rho' = (\pi/2) - \rho_o .$$

The system (4.74) actually has an infinite number of solutions. To find a few explicitly we look for a solution for which $\pi y > 5$ so that $\tanh \pi y = 1$. Thus division of the first of (4.74) by the second gives us

$$y \log d_o + \rho' = \pi(x+\Delta) + 2\pi r \tag{4.75}$$

where r is an integer. Then we substitute $\cosh \pi y = \sinh y = e^{\pi y}/2$ for r large enough and get

$$\pi y = \left(\frac{\pi^2}{\pi^2 + (\log d_o)^2}\right)\left\{\delta+\log 2 + \left(\log d_o\right)\left(\Delta+\tfrac{1}{2} - \frac{\rho'}{\pi} + 2r\right)\right\},$$

$$x = \left(\frac{\log d_o}{\pi^2 + (\log d_o)^2}\right)\left\{\left(\delta+ \log 2 + \tfrac{1}{2}\log d_o\right) - \frac{\pi^2}{\log d_o}\left(\Delta+2r- \frac{\rho'}{\pi}\right)\right\}.$$

Negative y's are also solutions. We need only one of the solutions to exhibit the root to the dispersion equation. Finally

$$\bar{s} = +i -\bar{\nu}\left((1+d_o)\mu^2 - \log \mu/\log d_o + (x^{\mathcal{O}} + iy)\right). \tag{4.77}$$

From the second of equations (4.51) one gets

$$\bar{s} = -i - \bar{\nu}((1+d_o)\mu^2 - \log\mu/\log d_o + (x'+\dot{(}y')) \ . \tag{4.78}$$

We cannot say that there are an infinite number of these resonances although there are an infinite number of solutions to (4.74). The reason is that $|h|$ must be smaller than $\log|\mu|$ for the analysis to work.

Should $|\tilde{\alpha}_1|$ be of order μ^2 one may no longer omit the term in $\bar{\Phi}$ since we know from equations (4.61) to (4.63) that to lowest order

$$|\bar{\Phi}| = \frac{\Theta}{\mu^2} \ , \tag{4.79}$$

where for $|\mu| >> 1$

$$(1 + d_o)^{-1} < \Theta < d_o^{-1}.$$

The analysis given here is good for $O|\tilde{\alpha}_{(1)}| < O(\mu^2)$ so that for $|\mu|$ sufficiently large one may ignore the contribution of $\bar{\Phi}$.

If we keep $|\mu|$ fixed and large as $\bar{\nu} \rightarrow 0$ there is one limiting root consistent with the dispersion equation (4.53). This root appears when $\bar{\omega}_o^2 \rightarrow 0$ with $\bar{\nu}$ in such a way that $\omega_o^2/\bar{\nu}$ remains fixed. The root is $\bar{s} = i$, and correspondingly there is a root $\bar{s} = -i$ from the second of equations (4.51). In

this limit $\overline{K}$ also goes to zero so that in effect we have computed one of the zero order resonances discussed by Liboff[2].

5. Electromagnetic Propagation and Plasma Oscillations

A. Introductory Comments

When $\underset{\sim}{k}$ is parallel to the applied magnetic field $\underset{\sim}{B}_o$ the dispersion equation (3.23) simplifies after a simple rearrangement of rows and columns to

$$D_+ \cdot D_- \cdot D_3 = 0. \tag{5.1}$$

In (5.1)

$$D_{\pm} = \begin{vmatrix} (-1+\alpha I_{\pm\pm}) & \beta I_{\pm\pm} \\ \hat{\beta}\hat{I}_{\pm\pm} & (-1+\hat{\alpha}\hat{I}_{\pm\pm}) \end{vmatrix}, \tag{5.2}$$

and

$$D_3 = \begin{vmatrix} D_{31} & D_{32} \\ \hat{D}_{32} & \hat{D}_{31} \end{vmatrix} \tag{5.3}$$

where

$$D_{31} = \begin{vmatrix} (-1+(\alpha-\eta)H_3) & (-iC(\nu-\tfrac{1}{2}\nu_2)H_T) \\ (i(\alpha-\eta)H_T) & C(3-(\nu-\tfrac{1}{2}\nu_2)I_{TT} \end{vmatrix},$$

and

$$D_{32} = \begin{vmatrix} (\beta+\eta)H_3 & -\,i\,\frac{C}{2}\,\nu_2\,H_T \\ i(\beta+\eta)H_T & -\,\frac{C}{2}\,\nu_2\,I_{TT} \end{vmatrix}.$$

The determinants $\hat{D}_{31}$ and $\hat{D}_{32}$ are obtained from D_{31} and D_{32} by placing carets over the appropriate quantities.

In section 5B we set $D_{\pm} = 0$ and compute the effect of a collisional plasma in a magnetic field on the vacuum electromagnetic roots $s^2+p^2 = 0$. To do this we suppose that $p = kc$ is larger than any frequency appearing in $D_{\pm}$. We then suppose that $s = ip(1+\Delta)$ where Δ is complex and $|\Delta| << 1$. All of this enables one to expand the integrals and the quantities α, $\hat{\alpha}$, η, $\hat{\eta}$ in terms of the frequencies made non-dimensional with respect to $|p|$. The ion mass remains finite in this computation.

In section 5C We set $D_3 = 0$ and compute the collisional correction to the plasma oscillation mode. Here the plasma frequency ω_o is chosen larger than any other frequency appearing in D_3 and again we suppose that $s = i\omega_o(1+\Delta)$ where Δ is complex and $|\Delta| << 1$. In the parameter range we have chosen the result for finite ion mass does not reduce when we set $\hat{m} \rightarrow \infty$ to the result of Lenard and Bernstein[6]. We give the reason for this disparity at the appropriate point. At the same time we compare results with those of the Krook model theories of Lewis and Keller and of Liboff.

B. Electromagnetic Propagation

The principal reason for performing this calculation was to check the applicability of a formula describing the Faraday rotation of a plane wave by a plasma in a steady magnetic field.

The formula, equation (5.6) below, was derived for propagation parallel to the steady magnetic field in a collisionless plasma from simplified macroscopic equations of motion. The experimental arrangement of Dr. Fred L. Ribe*, however, corresponds to the following parametrization. All frequencies are made non-dimensional with respect to $p = kc$, and since his source is an optical maser we take a representative wavelength to be of the order of 6×10^{-5} cm. The small parameter for this experimental set up is

$$\delta = (m/\hat{m}) \ll 1.$$

The collision frequencies as computed from equations (2.14) are

$$\begin{aligned} \varepsilon_1 &= \frac{\nu_1}{p} = O(\delta), \\ \hat{\varepsilon}_1 &= \frac{\hat{\nu}_1}{p} = O(\delta^{3/2}), \\ \varepsilon_2 &= \nu_z/p = O(\delta^2), \\ \varepsilon &= \varepsilon_1+\varepsilon_2 \quad , \quad \hat{\varepsilon} = \hat{\varepsilon}_1+\varepsilon_2 \quad . \end{aligned} \tag{5.4}$$

In addition

$$\begin{aligned} \tau_o &= \frac{\omega_o^2}{2p^2} = O(\delta) \ , \\ \frac{K^2}{p^2} &= \frac{C^2}{c^2} = O(\delta) \quad , \end{aligned} \tag{5.5}$$

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* Private communication.

and

$$\frac{|\Omega|}{p} = O(\delta) .$$

A plane wave traveling in the direction of $\underset{\sim}{B}_o$ may be resolved into two circularly polarized waves both of which satisfy the relation $s = ip$ in the absence of plasma. However, according to the macroscopic analysis of, for example, L. Spitzer[14] in the presence of plasma the two circularly polarized waves denoted by s_+ and s_- satisfy

$$s_{\pm} = ip \left\{ 1+\tau_o \left(1 - \frac{\tau_o}{2} \pm \frac{|\Omega|}{p}\right) \right\} . \tag{5.6}$$

To get (5.6) from Spitzer's equation (4-13) one expands his formula and keeps terms at most $O(\delta^2)$. The difference in frequency between the two polarized waves

$$\frac{s_+ - s_-}{i} = 2p\tau_o \frac{|\Omega|}{p} \tag{5.7}$$

is a possible diagnostic tool for measuring plasma densities since p and $|\Omega|$ are known and ω_o^2 is proportional to n_o. The question to be answered is: Do the formulas given in equation (5.6) and (5.7) change for the experimental parameters used by Dr. Ribe. These parameters are given in rough form by equations (5.4) and (5.5). From $D_+ = 0$ we find s_+ and from $D_- = 0$ we find s_-. For s_+ the dispersion equation can be written

$$D_+ = 0 = \begin{vmatrix} (-1 + (\frac{\alpha}{p})I_{++}) & (\frac{\beta}{p})I_{++} \\ (\frac{\hat{\beta}}{p})\hat{I}_{++} & (-1 + (\frac{\hat{\alpha}}{p})\hat{I}_{++}) \end{vmatrix} , \tag{5.8}$$

where

$$(\frac{\beta}{p}) = \frac{M}{m} (\varepsilon_2 + \frac{\omega_o^2 s}{p(s^2+p^2)}) ,$$

$$(\frac{\alpha}{p}) = \varepsilon - (\frac{\beta}{p}) ,$$

$$(\frac{\hat{\beta}}{p}) = \frac{m}{\hat{m}} (\frac{\beta}{p}) ,$$

$$(\frac{\hat{\alpha}}{p}) = \hat{\varepsilon} - (\frac{\hat{\beta}}{p}) ,$$

$$\omega_o^2 = \frac{e^2 n_o}{\varepsilon_o M} ,$$

$$M = \frac{m\hat{m}}{m+\hat{m}} .$$

$$I_{++} = \int_0^\infty dt \exp[- (\frac{s}{p} + \varepsilon + i \frac{\Omega}{p})t - \varepsilon\mu^2 t + \mu^2(1-e^{-\varepsilon t})] \tag{5.9}$$

where

$$\mu^2 = \frac{K^2}{p^2} \cdot \frac{1}{\varepsilon^2} .$$

We obtain $\hat{I}_{++}$ from I_{++} by replacing $(\varepsilon, \Omega, \mu^2)$ by $(\hat{\varepsilon}, \hat{\Omega}, \hat{\mu}^2)$. The integral (5.9) and the dispersion relation (5.8)

are the result of changing the variable of integration according to $y = t/p$.

To find the roots we guess that

$$\frac{s}{ip} = [1 + \tau_o(1+h_x + ih_y)] \tag{5.10}$$

where h_x and h_y are the unknown functions to be determined. We suppose that h_x and h_y are $O(\delta)$ and verify the guess at the end of the computation. One substitutes (5.10) into the formula for $(\frac{\beta}{p})$ given after equation (5.8). Then by the remaining equations $(\frac{\alpha}{p})$, $(\frac{\hat{\beta}}{p})$ and $(\frac{\hat{\alpha}}{p})$ are known. The result for $(\frac{\beta}{p})$ after expansion in the small parameters is to first order

$$(\frac{\beta}{p}) = \frac{M}{m}\,[(\varepsilon_2 - h_y) + i(-1 + h_x - \frac{\tau_o}{2})] \,. \tag{5.11}$$

The integral I_{++} given by (5.9) becomes

$$I_{++} = \int_o^\infty e^{-i[1+\tau_o(1+h_x)+\,\Omega/p\,-i(\varepsilon-\tau_o h_y)]t}\,\exp[-\varepsilon\mu^2 t+\mu^2(1-e^{-\varepsilon t})]. \tag{5.12}$$

After first expanding the argument of the second exponential to order δ^2 and then expanding the exponential itself one gets

$$I_{++} = \int_o^\infty dt\; e^{-i(1+\tau_o+h_x\tau_o+\,\Omega/p\,-\,i(\varepsilon-\tau_o h_y))t}.\left[1 - \frac{c^2}{2c^2}\,t^2 + \frac{\varepsilon c^2}{6c^2}\,t^3 + \frac{c^4 t^4}{8c^4} + \cdots\right] \tag{5.13}$$

We then integrate I_{++} according to the formula

$$\int_0^\infty e^{-at}\, t^n\, dt = \frac{n!}{a^{n+1}} \qquad \text{for real part } a > 0$$

which is satisfied if $\varepsilon - \tau_o h_y > 0$ as was guessed. The next step is to expand the terms

$$\left(\frac{1}{i(1+\tau_o(1+h_x) + \frac{\Omega}{p} - i(\varepsilon-\tau_o h_y))}\right)^N \quad \text{for } N = 1 \text{ and } 3 \text{ to}$$

lowest order and collect terms. We substitute the results for I_{++}, $\hat{I}_{++}$, $(\frac{\alpha}{p})$, $(\frac{\beta}{p})$, $(\frac{\hat{\alpha}}{p})$ and $(\frac{\hat{\beta}}{p})$ into (5.8), expand the determinant, and equate the real and the imaginary parts separately and obtain to lowest order

$$h_x = -\frac{\tau_o}{2} + \frac{C^2}{c^2} - \frac{\Omega}{p} \tag{5.14}$$

and

$$h_y = (\varepsilon_2 + 2\,\frac{C^2}{c^2}\,\varepsilon_1)\ .$$

From equation $D_- = 0$ we find that h_y is the same but that

$$h_x = -\frac{\tau_o}{2} + \frac{C^2}{c^2} + \frac{\Omega}{p}\ . \tag{5.15}$$

Finally, using $\Omega = -|\Omega|$, we obtain

$$\frac{S_\pm}{ip} = \left(1 + \frac{\omega_o^2}{2p^2}\left(1 - \frac{\omega_o^2}{4p^2} + \frac{C^2}{c^2} \pm \frac{|\Omega|}{p}\right)\right) + i\,\frac{\omega_o^2}{2p^2}\,h_y\ , \tag{5.16}$$

and

$$\frac{s_+ - s_-}{ip} = \frac{\omega_o^2}{p^2} \frac{|\Omega|}{p} . \tag{5.17}$$

Equation (5.17) is the same as equation (5.7). That $\frac{C^2}{c^2} = \frac{\kappa T_o}{mc^2} = O(\delta)$ corresponds to an equilibrium temperature of the order of 400 electron volts. At lower temperatures the effect of neutral particles would have to be included. Equation (5.17) would then measure the electron (and hence ion) density and not the neutral particle density. Equation (5.16) tells us that collisional damping appears to third order in the small parameter δ .

The solution corresponding to the vacuum root $s = -ip$ is obtained by taking the complex conjugates of equations (5.16) and (5.17).

C. Plasma Oscillations

We study plasma oscillations in a region that has been covered extensively in the literature. In this region $(\lambda d/\lambda) \ll 1$ and $(\lambda d/\lambda c) \ll 1$. The approach taken here may be compared with the approach of Lenard and Bernstein[6] who used a model most similar to ours in the absence of a magnetic field and the Krook theories of Lewis and Keller[19] and R. Liboff[2]. We show that the IFP results and Liboff's results are the same and that the results of Lenard and Bernstein are the same as the results of Lewis and

Keller. This means that the calculation is not sensitive to collision model but is sensitive to the way the equations are linearized in the chosen parameter range. We shall make the point more explicit.

To make the computation precise we choose

$$\frac{\nu_1}{|\omega_o|} = \varepsilon_1' \ll 1 \tag{5.18}$$

as the small parameter. Then

$$\frac{\nu_2}{|\omega_o|} = \varepsilon_2' , \qquad \hat{\varepsilon}_1' = \frac{\hat{\nu}_1}{|\omega_o|} , \tag{5.19}$$

$$\varepsilon' = \varepsilon_1' + \varepsilon_2' \qquad \text{and} \qquad \hat{\varepsilon} = \hat{\varepsilon}_1' + \varepsilon_2'$$

are all of order ε_1' or smaller by mass ratios. The mass ratios are kept independent of ε_1'. The remaining parameter $q_o = \lambda d/\lambda$ is chosen to satisfy

$$q_o^2 = (\lambda d/\lambda)^2 = (K^2/\omega_o^2) = O(\varepsilon_1') . \tag{5.20}$$

The first hypothesis is that for $\omega_o > 0$

$$s = i\omega_o(1+h_x'+h_y'i) = i\omega_o(1+\Delta'), \tag{5.21}$$

where h_x' and h_y' are supposed $O(\varepsilon_1')$.

The next hypothesis is that H_T, I_{TT}, $\hat{H}_T$ and I_{TT}, contribute terms of order higher than ε_1' to the equation. The first hypothesis is verified at the end of the calculation, the second, which one verifies by substitution of (5.21) into the matrix elements containing H_T, I_{TT}, $\hat{H}_T$ and $\hat{I}_{TT}$, is not given here. With the two hypothesis the dispersion equation $D_3 = 0$ may be replaced with

$$D_3' = \begin{vmatrix} \left(-1 + \left(\frac{\alpha-\eta}{\omega_o}\right) H_3\right) & \left(\frac{\beta+\eta}{\omega_o}\right) H_3 \\ \left(\frac{\hat{\beta}+\hat{\eta}}{\omega_o}\right) \hat{H}_3 & \left(-1 + \left(\frac{\hat{\alpha}-\hat{\eta}}{\omega_o}\right) \hat{H}_3\right) \end{vmatrix} = 0, \tag{5.22}$$

where

$$\left(\frac{\beta+\eta}{\omega_o}\right) = \frac{M}{m}\left(\varepsilon_2' + \frac{1}{i(1+\Delta)}\right),$$

$$\left(\frac{\alpha-\eta}{\omega_o}\right) = \left(\varepsilon' - \frac{M}{m}\varepsilon_2'\right) + \frac{iM}{m(1+\Delta)},$$

$$\left(\frac{\hat{\beta}+\hat{\eta}}{\omega_o}\right) = \frac{M}{\hat{m}}\left(\varepsilon_2' + \frac{1}{i(1+\Delta)}\right),$$

$$\left(\frac{\hat{\alpha}-\hat{\eta}}{\omega_o}\right) = \left(\hat{\varepsilon}' - \frac{M}{\hat{m}}\varepsilon_2'\right) + \frac{iM}{\hat{m}(1+\Delta)},$$

$$H_3 = \frac{s}{\varepsilon'\omega_o}\int_0^\infty e^{\phi}(1-e^{-\varepsilon' t})dt,$$

where

$$\phi = -\left(\frac{s}{\omega_o}\right)t - \varepsilon'\mu^2 t + \mu^2(1-e^{-\varepsilon' t}).$$

We expand H_3 in terms of the small parameter ε' and use the identity

$$q_o^2 = \mu^2\varepsilon'^2 = (K^2/\omega_o^2).$$

The result is

$$H_3 = i(1+\Delta)\int_0^\infty e^{-i(1+\Delta)t - q_o^2 t^2/2}\,(t - t^2\,\varepsilon'/2)\,dt\ . \tag{5.23}$$

With the substitution $y = q_o t$ and $\sigma = (1+\Delta)/q_o$ we get

$$H_3 = i\sigma\int_0^\infty e^{-i\sigma y - y^2/2}\,dy\left[\frac{1}{q_o}\,y - \frac{\varepsilon'}{2q_o^2}\,y^2\right]\ . \tag{5.24}$$

The integral H_3 is related to the integral $I_o = \int_0^\infty e^{-i\sigma y - y^2/2}dy$ by use of the identity

$$ye^{\phi_o} = \left(-i\sigma e^{\phi_o} - \frac{de^{\phi_o}}{dy}\right) \tag{5.25}$$

where

$$\phi_o = -i\sigma y - y^2/2\ .$$

The result is

$$I_1 = \int_0^\infty ye^{\phi_o}dy = 1 - i\sigma I_o\ ,$$

$$\tag{5.26}$$

and similarly

$$I_2 = \int_0^\infty y^2 e^{\phi_o}dy = I_o - i\sigma I_1\ .$$

Evaluation of I_o is effected for $|\sigma| \gg 1$ by means of the formula in appendix B of reference [19], after a trivial change in variables. One may also obtain this formula by

integrating by parts a sufficient number of times. The result is

$$I_o = \int e^{-i\sigma y - y^2/2} dy \sim \frac{1}{i\sigma} \left(1 + \frac{1}{\sigma^2} + \frac{3}{\sigma^4} + O\left(\frac{1}{\sigma^6}\right)\right) . \tag{5.27}$$

Then

$$\begin{aligned} I_1 &\sim -\frac{1}{\sigma^2} \left(1 + \frac{3}{\sigma^2} + O\left(\frac{1}{\sigma^4}\right)\right) , \\ I_2 &\sim -\frac{2}{i\sigma^3} \left(1 + O\left(\frac{1}{\sigma^2}\right)\right) . \end{aligned} \tag{5.28}$$

Substitution of (5.28) into (5.24) gives

$$H_3 = -\frac{i}{1+\Delta} (1+3q_o^2) + \varepsilon' . \tag{5.29}$$

to lowest order. We substitute (5.29), the corresponding value for $\hat{H}_3$ (gotten by replacing ε' with $\hat{\varepsilon}'$) and the quantities $\left(\frac{\alpha-\eta}{\omega_o}\right)$, $\left(\frac{\hat{\alpha}-\hat{\eta}}{\omega_o}\right)$, $\left(\frac{\beta+\eta}{\omega_o}\right)$, $\left(\frac{\hat{\beta}+\hat{\eta}}{\omega_o}\right)$ back into (5.22); expand to lowest order in ε'_1; and equate the real and imaginary terms separately to zero. The result to first order is

$$\begin{aligned} h_x &= \frac{3q_o^2}{2} , \\ h_y &= \frac{\varepsilon_2'}{2} , \end{aligned} \tag{5.30}$$

or

$$s = i\omega_o(1 + 3K^2/2\omega_o^{\ 2}) - \nu_2/2 \quad . \tag{5.31}$$

The complex conjugate of (5.31) is also a solution.

In the limit $\hat{m} \longrightarrow \infty$ the dispersion equation (5.22) reduces to

$$1 = (\nu_1 - \frac{\omega_o^2}{s}) \frac{s}{\nu_1} \int_o^\infty e^{\phi} dy(1-e^{-\nu_1 y})dy \tag{5.32}$$

where

$$\phi = - \left[s + \frac{K^2}{\nu_1} \right] y + \frac{K^2}{\nu_1^2} (1-e^{-\nu_1 y}) \ .$$

If we compute only to first order by the same techniques leading to (5.31) we get

$$s = \pm\ i\omega_o(1 + 3K^2/2\omega_o^2) \tag{5.33}$$

with no damping.

To compare with Lenard and Bernstein[6] we observe that if $\hat{m} \longrightarrow \infty$, if $\underset{\sim}{B}_o = 0$, if the temperature terms are neglected, and if one imposes the contraint that $\underset{\sim}{E}$ be parallel to $\underset{\sim}{k}$, then the dispersion equation (3.23) reduces to (5.32) in which the direction of $\underset{\sim}{k}$ is now arbitrary. The difference between their dispersion equation and (5.32) is that instead of the term $(\nu_1 - \omega_o^2/s)$ in (5.32) they have $(- \omega_o^2/s)$. The missing term ν_1 stems from the relaxation effect of the perturbed velocity on the

perturbed distribution function in the linearized Boltzmann's equation (2.13). The missing term is the factor $\nu_1 \underset{\sim}{u}$ in the quantity

$$\underset{\sim}{a} = e/m\, \underset{\sim}{E} + \nu_1 \underset{\sim}{u}$$

given in equation (2.13) (in the limit $\hat{m} \rightarrow \infty$). The term $\nu_1 \underset{\sim}{u}$ appears when one linearizes the collision terms which give the macroscopic conservation equations. The theory of Lenard and Bernstein does not correspond to such a linearization. The consequence of omitting the term $\nu_1 \underset{\sim}{u}$ is that the root of the dispersion formula is (after one corrects their equation (38) for a typographical error) in our notation

$$s = \pm\, i\omega_o(1 + 3K^2/2\omega_o^2) - \nu_1/2 \,. \tag{5.34}$$

One may write the linearized longitudinal dispersion equation of Lewis and Keller (equation (5.7) reference [19]) in our notation as

$$[\omega_o^2(s+\nu_1)+K^2\nu_1][1-(s+\nu_1)\int_0^\infty e^{\phi'}\,dy] = -sK^2 \,, \tag{5.35}$$

where

$$\phi' = -(s+\nu_1)y - \frac{K^2y^2}{2} \,.$$

Equation (5.35) is the Krook model version of the dispersion equation of Lenard and Bernstein. The result of computing the

root corresponding to the plasma oscillation is precisely equation (5.34). This reflects the fact that the linearized Boltzmann's equation of Lewis and Keller lacks the same term as does the equation of Lenard and Bernstein. If one computes from Liboff's equations[2], on the other hand, one should obtain our equation (5.33). This reflects the fact that Liboff's equations mimic the Boltzmann's equations and when linearized contain a term like $\nu_1 \underset{\sim}{u}$.

In the chosen parameter region the mechanism of damping to first order is given by the ion-electron collision frequency ν_2 as seen in equation (5.31). A properly linearized Krook theory should give the same answer in the same parameter region when the collision frequencies are suitably identified.

Equation (5.31) agrees with the results of Johnston and Shkarofsky[21] to the order given. Their results are computed from a set of linearized macroscopic equations obtained by estimating collisional effects from the original Fokker-Planck coefficients to lowest order.

Their equations also satisfy conservation of mass, momentum and energy for finite ion mass. It is therefore not surprising that to the order of equation (5.31) the results agree.

6. Summary and Conclusions

The actual Boltzmann's and Fokker-Planck equations are too complicated at present to be of use in constructing a dispersion equation. For this reason one seeks collision models which retain as many of the properties of the original equations as possible. There are two models which retain the property of giving the macroscopic conservation equations: Liboff's generalization of the Krook model[2] and the isotropic Fokker-Planck model introduced here. Of the two we expect that the IFP model is superior since it retains the physical property of describing a diffusion process in velocity space. We have, therefore, increased confidence in those results which are the same in both theories and where there is disagreement we expect the IFP results to be better.

In general one expects that long wavelength disturbances behave macroscopically. For these disturbances, the magnetohydrodynamic and plasma electromagnetic modes, there is little difference in the models. Differences appear as small higher order corrections. In section 5 where the Faraday rotation and plasma oscillations are studied the two theories give the same corrections since the collision frequencies are chosen to be small with respect to the representative frequencies.

The greatest disparity between the Krook and IFP models appears in the results for the microscopic Larmor resonances in the long wavelength limit. We see in sections 4B and 4D that the Krook model does not describe a diffusion process in velocity

space since it gives essentially the same results as the collisionless model. Where the Krook model predicts an infinite number of Larmor resonance modes only at propagation perpendicular to the applied magnetic field the IFP model predicts an infinite number of Larmor resonances at arbitrary directions of propagation (except parallel to the applied field). At perpendicular propagation, for each value of the integer N multiplying the Larmor frequency there is but one Krook resonance and an infinite sequence of IFP resonances differing by integral multiples of the collision frequency. This last difference reflects the mathematical property that the IFP differential operator has an unbounded discrete spectrum. The unboundedness of the spectrum is reflected again at parallel propagation where there are at the Larmor frequency $(|N| = 1)$ an infinite number of resonances which differ by integral multiples of the collision frequency. Such modes are not found from the Krook theory. Here the mean free path was small with respect to the wavelength. For mean free path large with respect to wavelength we find in section 4F strongly damped resonances at the Larmor frequency again at parallel propagation.

In section 4C we study the new Larmor resonance modes at arbitrary angles of propagation in the non-uniform collisionless limit obtained by keeping the ratio of mean free path to wavelength fixed and small as the collision frequencies become small. We find that the new Larmor resonance modes remain in this limit. These results indicate the possibility of experimental verification

if one viewed radiation from a plasma at angles other than perpendicular to the applied magnetic field. They are, however, at present only suggestive since we have not examined the mechanism for exciting such modes.

S. Tanaka et al.[22] have studied microwave radiations from a dc discharge plasma in a magnetic field. Although their measurements appear to be of Larmor resonances their results are not sensitive enough to distinguish between models, especially since they view the radiation at a direction perpendicular to the applied magnetic field. Their measurements do agree with the rough observation that for a denser plasma the Larmor resonances are given by $s = iN|\Omega| + \cdots$ where $|N| > 1$ and $N = \pm 1$ is not a root of the dispersion equation. As the density becomes small the roots $s = \pm i|\Omega|$ are approached. This observation is contained in Liboff's study[2] since those roots which approach $s = \pm i|\Omega|$ as ω_o^2 and ν both become small linearly with the density are among his lowest order plasma electromagnetic modes.

Appendix A

List of Integrals:

$$I_{33} = \int_0^\infty e^{\phi} \left\{ e^{-\nu y} - \mu_3^2 (1-e^{-\nu y})^2 \right\} dy \quad ,$$

$$I_{3\pm} = \frac{-K^2(\nu \mp i\Omega)}{\nu(\nu^2+\Omega^2)} \int_0^\infty e^{\phi} (1-e^{-\nu y})(i-e^{-(\nu \pm i\Omega)y}) dy \ ,$$

$$I_{\pm\pm} = \int_0^\infty e^{\phi} \left\{ e^{-(\nu \pm i\Omega)y} + \frac{\mu_\perp^2}{2} e^{\pm i\psi} (1-e^{-(\nu \pm i\Omega)y})^2 \right\} dy \quad ,$$

$$I_{+-} = -\frac{K^2}{\nu^2+\Omega^2} \int_0^\infty e^{\phi} (1-e^{-(\nu+i\Omega)y}(1-e^{-(\nu-i\Omega)y}) dy \quad ,$$

$$H_3 = I_{33} + \frac{\sin^2\theta}{2} (I_{3+}+I_{3-}) = \frac{s}{\nu} \int_0^\infty e^{\phi} (1-e^{-\nu y}) dy \quad ,$$

$$H_\pm = I_{\pm\pm} + \cos^2\theta I_{3\pm} + \frac{\sin^2\theta}{2} I_{+-} = \frac{s}{\nu \pm i\Omega} \int_0^\infty e^{\phi} (i-e^{-(\nu \pm i\Omega)y}) dy \quad ,$$

$$H_o = H_3 \cos^2\theta + \frac{\sin^2\theta}{2} (H_+ + H_-) = \frac{s}{K^2} (1 - s \int_0^\infty e^{\phi} dy) \quad ,$$

$$I_{3T} = \frac{K}{\nu}\int_0^\infty e^{\phi}\left\{-2e^{-\nu y}(1-e^{-\nu y}) + Q_1(1-e^{-\nu y})\right\}dy \quad ,$$

$$I_{\pm T} = \frac{K(\nu\pm i\Omega)}{\nu^2+\Omega^2}\int_0^\infty e^{\phi}\left\{-2e^{-(\nu\pm i\Omega)y}(1-e^{-(\nu\mp i\Omega)y}) - e^{\pm i\psi}Q_1(1-e^{-(\nu\pm i\Omega)y})\right\}dy \quad ,$$

$$I_{TT} = \int_0^\infty e^{\phi}\left\{6e^{-2\nu y} - 4\mu_3^2 Q_3 + 2\mu_\perp^2 Q_\perp + Q_1^2\right\} \quad ,$$

$$H_T = I_{3T}\cos^2\theta + \frac{\sin^2\theta}{2}(I_{+T}+I_{-T}) = -\frac{s}{K}\int_0^\infty e^{\phi}Q_1 dy \quad ,$$

$$\mu^2 = \mu_3^2 + \mu_\perp^2 \quad ,$$

$$\mu_3^2 = \frac{K^2\cos^2\theta}{\nu^2} \quad , \quad \mu_\perp^2 = \frac{K^2\sin^2\theta}{\nu^2+\Omega^2} \quad , \quad K^2 = k^2C^2 \quad ;$$

$$\phi = -(s+\nu\mu^2)y + \mu_3^2(1-e^{-\nu y}) + \mu_\perp^2(-\cos\psi + \frac{e}{2}^{i\psi-(\nu+i\Omega)y} + e^{-i\psi-(\nu-i\Omega)y}),$$

$$e^{\pm i\psi} = -\frac{(\nu\mp i\Omega)^2}{\nu^2+\Omega^2} \quad ;$$

$$Q_1 = \mu^2(1-e^{-\nu y})^2 - \mu_\perp^2 e^{-\nu y}(e^{i\Omega y/2} - e^{-i\Omega y/2})^2 \quad ,$$

$$Q_3 = e^{-\nu y}(1-e^{-\nu y})^2 \quad ,$$

$$Q_\perp = e^{-\nu y}\left\{e^{i(\psi+\Omega y)}(1-e^{-(\nu+i\Omega)})^2 + e^{-i(\psi+\Omega y)}(1-e^{-(\nu-i\Omega)y})^2\right\} .$$

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